

## Higher genus partition functions of meromorphic conformal field theories

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# Higher genus partition functions of meromorphic conformal field theories

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**ABSTRACT:** It is shown that the higher genus vacuum amplitudes of a meromorphic conformal field theory determine the affine symmetry of the theory uniquely, and we give arguments that suggest that also the representation content with respect to this affine symmetry is specified, up to automorphisms of the finite Lie algebra. We illustrate our findings with the self-dual theories at  $c = 16$  and  $c = 24$ ; in particular, we give an elementary argument that shows that the vacuum amplitudes of the  $E_8 \times E_8$  theory and the  $\text{Spin}(32)/\mathbb{Z}_2$  theory differ at genus  $g = 5$ . The fact that the discrepancy only arises at rather high genus is a consequence of the modular properties of higher genus amplitudes at small central charges. In fact, we show that for  $c \leq 24$  the genus one partition function specifies already the partition functions up to  $g \leq 4$  uniquely. Finally we explain how our results generalise to non-meromorphic conformal field theories.

**KEYWORDS:** Conformal and W Symmetry, Conformal Field Models in String Theory

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## 1 Introduction

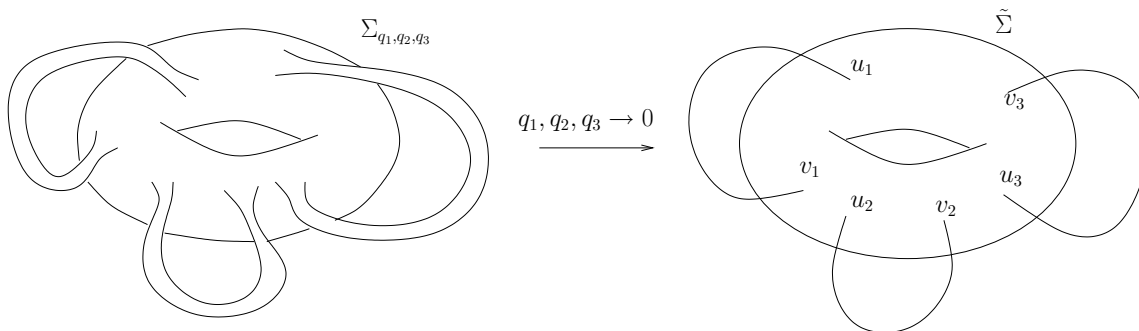
The genus one partition function of a conformal field theory determines the spectrum of the theory uniquely, but there are different conformal field theories that have the same genus one partition function. Probably the best known example is the case of the  $E_8 \times E_8$  and the  $\text{Spin}(32)/\mathbb{Z}_2$  theories at  $c = 16$  that have the same torus vacuum amplitude (and hence the same number of states at each conformal weight), but that are evidently different conformal field theories (since they have different Lie symmetries and thus have different correlation functions).

In the context of string theory, for example in the framework of the  $\text{AdS}_3/\text{CFT}_2$  correspondence [1, 2] (see also [3–5]) one often does not have direct access to the correlation functions of the (dual) conformal field theory that would specify the conformal field theory completely. Instead one has control over the vacuum amplitudes at arbitrary genus. It is then a natural question to ask to which extent this information specifies the (dual) conformal field theory uniquely.

In this paper we shall study this question for the case of meromorphic conformal field theories (that are relevant in the context of [1, 2]); the restriction to meromorphic theories simplifies our arguments, but is not crucial for our analysis, and essentially all our arguments work equally well in the general case. As we shall see, the higher genus vacuum amplitudes always determine the Lie symmetry of the theory completely, and we shall give arguments that suggest that the same is true for the representation content (up to automorphisms of the Lie algebra). As a special case, we give an elementary argument to show that the  $E_8 \times E_8$  and the  $\text{Spin}(32)/\mathbb{Z}_2$  theories have different genus  $g = 5$  amplitudes, in agreement with the recent analysis of [6].

The basic strategy of our analysis is as follows. There is a degeneration limit of a genus  $g$  surface in which it becomes a torus with  $g - 1$  nodes (see figure 1).

In this limit, the genus  $g$  vacuum amplitude is described by sums of  $2(g - 1)$  point functions on the torus, where we sum over an orthonormal basis of states inserted at  $(v_i, u_i)$ ,  $i = 1, \dots, g - 1$ , and weight the contribution of the state with conformal dimension  $h_i$  at  $(v_i, u_i)$  with  $q_i^{h_i}$ . If we consider the term that is proportional to  $\prod q_i$ , we get a sum over  $2(g - 1)$  point functions of currents (fields of conformal weight one). By integrating these currents along one of the cycles of the torus, we can convert them into zero modes. Thus, starting from a genus  $g$  vacuum amplitude, we can determine the trace over the full space of states, where we insert in addition to  $q^{L_0}$  also suitable combinations of generators of the finite dimensional Lie algebra. In fact, these combinations always define Casimir operators of the Lie algebra, and we can determine their eigenvalues (on the states of



**Figure 1.** A family of Riemann surfaces  $\Sigma_{q_1, q_2, q_3}$  of genus 4 degenerates to a singular surface with three nodes.

a given conformal dimension) from these considerations. This allows us to determine the underlying Lie algebra completely. We also argue, by considering more general degeneration limits, that we can determine the representation content of the theory (up to the ambiguity of the action of outer automorphisms) at arbitrary conformal weight.

We illustrate our findings with a number of explicit examples. In particular, we study the self-dual conformal field theories at  $c = 16$  and  $c = 24$  [7, 8], and show that all pairs of theories that have the same genus one partition function can be distinguished by their genus  $g = 5$  amplitudes. We also show that for  $c = 32$  such pairs of theories can typically already be distinguished at genus  $g = 2$ , and we give an explanation of these phenomena by studying the constraints from modularity systematically. Among other things, this allows us to show that for  $c \leq 24$  the genus  $g$  amplitudes with  $g \leq 4$  are already uniquely determined in terms of the genus  $g = 1$  amplitude, while no such constraint exists at  $c \geq 32$ .

Higher genus (vacuum) amplitudes of conformal field theories have been studied before among others in [9–15]. There is also some extended literature on higher genus amplitudes in string theory, see for example [16–18] and the reviews [19–21]; some more recent progress is also described in [6, 22–25].

The paper is organised as follows. In section 2 we outline the general structure of our argument. To illustrate the basic ideas we consider, in section 3, the examples of the self-dual meromorphic fields theories at  $c = 16$ , 24 and  $c = 32$ . In particular, we demonstrate that all pairs of inequivalent theories can be distinguished by (higher) genus amplitudes. In section 4 we analyse the modular properties of the higher genus amplitudes systematically, and thus explain our findings of section 3 from this perspective. In section 5 we work out the general argument that shows that higher genus amplitudes determine the Lie symmetry uniquely. We also argue, using similar techniques, that the same can be said about the representation content with respect to the affine algebras. (This result relies on a Lie algebraic conjecture for which we give some evidence in appendix C.) Finally, section 6 contains our conclusions where we indicate among other things how our arguments generalise to non-meromorphic conformal field theories. Appendix A gives some details of our calculations for  $c = 24$ , while appendix B collects some general facts about Riemann surfaces and their Schottky covers.

## 2 Partition functions and Lie algebra invariants

Let us begin by reviewing some standard material concerning genus  $g$  partition functions.

### 2.1 Partition functions and degeneration limits

In this paper we shall consider self-dual meromorphic conformal field theories, i.e. theories that are purely left-moving. These theories arise, for example, as the left-moving part of a holomorphically factorising conformal field theory, as in [1]. As we have mentioned before (see also the conclusions), our arguments also work for more general conformal field theories, but the restriction to meromorphic conformal field theories will simplify some of our notation considerably.

We shall always assume that the theory has a unique vacuum state  $\Omega$  of conformal dimension zero, and that the spectrum of  $L_0$  is a subset of the non-negative integers. This allows us to write

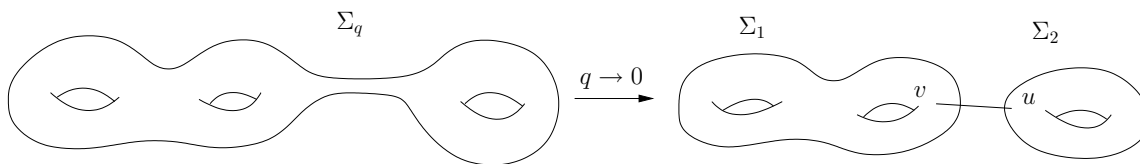
$$\mathcal{H} = \bigoplus_{h=0}^{\infty} \mathcal{H}_h, \tag{2.1}$$

where  $\mathcal{H}_h$  is the subspace of states of  $L_0$  eigenvalue  $h$ . We shall always assume that each eigenspace  $\mathcal{H}_h$  is finite dimensional. The genus one partition function of the theory then equals the genus one character  $\chi_{g=1}(\tau)$ , which is a holomorphic function of the modulus  $\tau$  of the torus. The usual modular consistency condition requires that  $\chi_{g=1}(\tau)$  is modular invariant; if we think of the meromorphic conformal field theory to be the left-moving part of a holomorphically factorising theory, the character only has to be modular invariant up to a phase. In either case, the modular  $S$ -matrix is essentially trivial, and hence Verlinde's formula implies that the meromorphic conformal field theory has only one representation, namely  $\mathcal{H}$  itself.

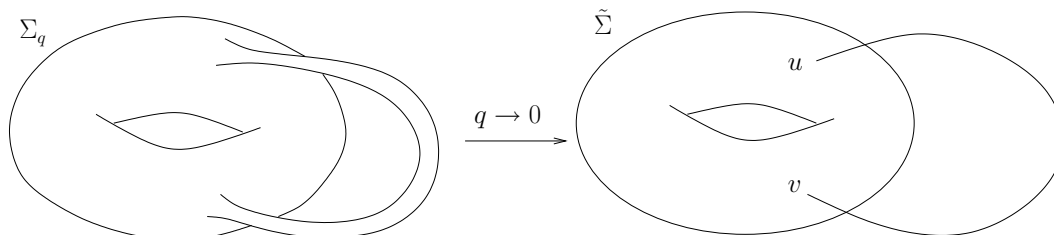
The genus  $g$  analogue of the chiral character  $\chi_{g=1}(\tau)$  defines a holomorphic section  $\chi_g$  in the line bundle  $\lambda^{c/2}$  that is defined on the moduli space  $\mathcal{M}_g$  of Riemann surfaces of genus  $g$ . Here  $\lambda$  is the determinant line bundle and  $c$  the central charge (see, for example, [11] for details). Again, for the left-moving part of a holomorphically factorising theory, the genus  $g$  partition function  $\chi_g$  must satisfy appropriate modular properties under  $\text{Sp}(2g, \mathbb{Z})$ ; this will be described in more detail in section 4.

The genus  $g$  partition function  $\chi_g$  also satisfies certain factorisation relations. Let  $\Sigma_q$  be a family of Riemann surfaces that degenerate in the limit  $q \rightarrow 0$ . There are two cases of interest: first, a homologically trivial cycle can be pinched down to a node. In this case the limit  $q \rightarrow 0$  describes a union of two connected components,  $\Sigma_1$  and  $\Sigma_2$  of genus  $k$  and  $g - k$ ,  $1 \leq k \leq [g/2]$ , respectively (see figure 2). The other case is that a homologically non-trivial cycle is pinched down, in which case the degenerate limit surface has genus  $g - 1$  (see figure 3). In either case, the genus  $g$  partition function converges to the partition function of the appropriate limiting surface. For example, in the second case where a homologically non-trivial cycle is pinched, the partition function becomes

$$\chi_g(\Sigma_q) \xrightarrow{q \rightarrow 0} \chi_{g-1}(\tilde{\Sigma}), \tag{2.2}$$



**Figure 2.** By a separating degeneration limit of a family of smooth Riemann surfaces, a singular Riemann surface with node (here represented by a line) is obtained. The surface is given by two smooth components  $\Sigma_1$  and  $\Sigma_2$  of genus  $k$  and  $g - k$  with marked points  $u \in \Sigma_1$  and  $v \in \Sigma_2$  joined by a node.



**Figure 3.** A singular Riemann surface obtained by a non-separating degeneration limit. The points  $u, v$  on a surface  $\tilde{\Sigma}$  of genus  $g - 1$  are identified to form a node (here represented by a line).

where  $\tilde{\Sigma}$  is the surface obtained from the singular curve by removing the node. The overall normalisation of the partition functions is fixed by  $\chi_{g=0} = 1$ .

Equation (2.2) describes the leading behaviour as  $q \rightarrow 0$ , but one can also be more specific about the subleading terms. In fact, in any such degeneration limit, the chiral partition function  $\chi_g$  can be expanded in a power series in the degeneration parameter  $q$  (see [9])

$$\chi_g = \sum_{h=0}^{\infty} q^h \sum_{i \in I_h} \left\langle V(\psi_i^{(h)}, u) V(\psi_i^{(h)}, v) \right\rangle_{\tilde{\Sigma}},$$

where  $h$  labels the eigenvalues of the  $L_0$  operator (conformal weights) in  $\mathcal{H}$ , and the  $\psi_i^{(h)}$ ,  $i \in I_h$  are an orthonormal basis for the states  $\mathcal{H}_h$  of conformal weight  $h$ .<sup>1</sup> Furthermore,  $V(\phi, z)$  denotes the vertex operator corresponding to the state  $\phi$ , and  $u, v \in \tilde{\Sigma}$  are the points on the (possibly disconnected) Riemann surface  $\tilde{\Sigma}$  that are identified by the node to form the singular surface  $\Sigma_0$ .

In the following we shall be interested in the particular case of multiple degenerations in which a Riemann surface of genus  $g$  becomes a surface of genus 1 with  $g - 1$  nodes (see figure 1 in the introduction). In this case it is useful to regard the partition function as a holomorphic function on the Schottky space [10] (see appendix B.3 for more details). The degeneration limit we are considering corresponds to the limit in which  $g - 1$  out of  $g$  multipliers  $q_1, \dots, q_{g-1}$  of the Schottky group generators vanish, so that, upon setting  $q \equiv q_g$ , we obtain

$$\chi_g = \sum_{h_1, \dots, h_{g-1}} \prod_{j=1}^{g-1} q_j^{h_j} \sum_{i_1, \dots, i_{g-1}} \text{Tr} \left( \prod_{j=1}^{g-1} V(\psi_{i_j}^{(h_j)}, u_j) V(\psi_{i_j}^{(h_j)}, v_j) q^{L_0} \right). \quad (2.3)$$

<sup>1</sup>The power series on the right hand side converges for sufficiently small  $q$ .

Note that the standard definition of the genus 1 character as the trace of the operator  $q^{L_0 - c/24}$  is related to  $\chi_1(q)$  as

$$\chi_1(q) = q^{c/24} \text{Tr} \left( q^{L_0 - c/24} \right) . \quad (2.4)$$

The extra factor  $q^{c/24}$  is due to the conformal transformation (see (B.13)) from the cylinder to the annulus. With this definition,  $\chi_1$  is smooth in the limit  $q \rightarrow 0$ , which corresponds to the degeneration of a torus to a sphere.

## 2.2 Lie algebra considerations

In the following we shall mainly be interested in the contribution to (2.3) from states at  $h_j = 1$ . We therefore need to review what is known about these states in general.

In any meromorphic conformal field theory, the states at conformal weight  $h = 1$  give rise to an affine Lie algebra symmetry (see for example [26] for a more detailed exposition). Indeed if we denote the fields of conformal dimension one (the currents) by  $J^a(z)$ , then their operator product expansion is necessarily of the form

$$J^a(z) J^b(w) = \frac{\kappa^{ab}}{(z-w)^2} + f^ab_c \frac{J^c(w)}{z-w} + \mathcal{O}(1), \quad (2.5)$$

where  $\kappa^{ab}$  and  $f^ab_c$  are constants. Defining the modes of these fields via

$$J_n^a = \oint dz z^n J^a(z), \quad (2.6)$$

it follows from (2.5) that they satisfy the commutation relations of an affine Kac-Moody algebra  $\hat{\mathfrak{g}}$

$$[J_m^a, J_n^b] = m \kappa^{ab} \delta_{m,-n} + f^ab_c J_{m+n}^c . \quad (2.7)$$

Note that the zero modes  $J_0^a \equiv t^a$  form a finite-dimensional Lie algebra  $\mathfrak{g}$  whose structure constants are given by  $f^ab_c$ . Furthermore,  $\kappa^{ab}$  is a symmetric tensor that is invariant with respect to  $\mathfrak{g}$ . If the conformal field theory is unitary, then  $\kappa^{ab}$  is positive definite, and thus the finite dimensional Lie algebra  $\mathfrak{g}$  is semi-simple, or a direct sum of simple Lie algebras and some  $u(1)$  factors.

In each simple factor,  $\kappa$  is proportional to the Cartan-Killing form  $K^{ab}$  of the Lie algebra  $\mathfrak{g}$ . We choose the standard convention for the normalisation of the Cartan-Killing form, namely that the longest roots of the Lie algebra have length squared equal to 2. Furthermore, we pick a basis for the Lie generators of  $\mathfrak{g}$  such that  $K^{ab} = \delta^{ab}$ . With these conventions  $\kappa^{ab}$  is then of the form

$$\kappa^{ab} = k \delta^{ab}, \quad (2.8)$$

where  $k$  is the level that takes a specific fixed value for each simple factor. If we assume that the theory is unitary then each  $k$  must be a positive integer. The coefficient of the identity in the OPE (2.5) determines the normalisation of the currents; at level  $k$ , the



currents  $J^a$  have therefore norm  $k$ . In order to have an orthonormal basis we therefore have to rescale them as

$$\hat{J}^a = k^{-\frac{1}{2}} J^a . \tag{2.9}$$

In the following the quadratic Casimir operator of the finite dimensional Lie algebra will play an important role. We choose the (usual) convention that the quadratic Casimir  $C_2$  is given by

$$C_2 = \sum_a t^a t^a . \tag{2.10}$$

In the adjoint representation the value of  $C_2$  is then equal to  $2h^\vee(\mathfrak{g})$ , where  $h^\vee(\mathfrak{g})$  is the dual Coxeter number of the finite dimensional Lie algebra  $\mathfrak{g}$ , and for the simply-laced algebras we have

$$h^\vee(a(n)) = n + 1, \quad h^\vee(d(n)) = 2n - 2, \quad h^\vee(e6) = 12, \quad h^\vee(e7) = 18, \quad h^\vee(e8) = 30 . \tag{2.11}$$

For the rescaled  $\hat{J}^a$  generators it then follows that

$$\text{Tr}_{ad}(\hat{J}_0^a \hat{J}_0^a) = \frac{2h^\vee(\mathfrak{g})}{k} \dim(\mathfrak{g}) . \tag{2.12}$$

### 2.3 Lie algebra invariants in degeneration limits

After this interlude we are ready to return to the degeneration limits of genus  $g$  partition functions. Let us consider the coefficients of (2.3) that contain at most linear powers of  $q_i$ , i.e.

$$\begin{aligned} \chi_g = & \text{Tr}(q^{L_0}) + \sum_{i=1}^{g-1} q_i \sum_a \text{Tr}(\hat{J}^a(u_i) \hat{J}^a(v_i) q^{L_0}) \\ & + \sum_{i \neq j} q_i q_j \sum_{a,b} \text{Tr}(\hat{J}^a(u_i) \hat{J}^a(v_i) \hat{J}^b(u_j) \hat{J}^b(v_j) q^{L_0}) \\ & + \dots + q_1 \dots q_{g-1} \sum_{a_1, \dots, a_{g-1}} \text{Tr} \left( \prod_{i=1}^{g-1} \hat{J}^{a_i}(u_i) \hat{J}^{a_i}(v_i) q^{L_0} \right) + \mathcal{O}(q_i^2), \end{aligned} \tag{2.13}$$

where  $\mathcal{O}(q_i^2)$  is a term of order 2 in at least one of the parameters  $q_1, \dots, q_{g-1}$ . The functions that appear on the right hand side are correlation functions of currents

$$\sum_{a_1, \dots, a_l} \text{Tr} \left( \prod_{i=1}^l \hat{J}^{a_i}(u_i) \hat{J}^{a_i}(v_i) q^{L_0} \right) . \tag{2.14}$$

If we know the vacuum amplitude at genus  $g$ , we can thus determine all these correlation functions, where the number of currents,  $2l$ , is less or equal than  $2(g-1)$  (and the modular parameter of the torus  $\tau$  is arbitrary). These amplitudes depend obviously on the Lie group symmetries of the theory, as well as its representations content. The simplest way to make this dependence explicit is to integrate the insertion points  $u_1, v_1, \dots, u_l, v_l$  along the  $\alpha$ -cycle of the torus. Because of (2.6) this then replaces the current  $\hat{J}^a$  by its zero mode,  $\hat{J}_0^a$ . In doing these integrals, there is a choice corresponding to the ordering of the integrals.

Thus we may take the  $2l$  zero modes to appear in any order. The simplest ordering is the one where the two zero modes  $\hat{J}_0^{a_i}$  stand next to each other, i.e. the term of the form

$$\sum_{a_1, \dots, a_l} \text{Tr} \left( \hat{J}_0^{a_1} \hat{J}_0^{a_1} \dots \hat{J}_0^{a_l} \hat{J}_0^{a_l} q^{L_0} \right) = \frac{1}{k^l} \text{Tr} \left( C_2^l q^{L_0} \right). \quad (2.15)$$

Since  $\hat{J}_0^a$  commutes with  $L_0$ , the coefficient of  $q^n$  in this series comes from the states of conformal weight  $n$ ,  $\mathcal{H}_n$ . Let us decompose  $\mathcal{H}_n$  in terms of irreducible representations of  $\mathfrak{g}$  as

$$\mathcal{H}_n = \bigoplus_R m_{n,R} R, \quad (2.16)$$

where  $m_{n,R}$  is the multiplicity with which the irreducible representation  $R$  appears in  $\mathcal{H}_n$ . If we denote the value of the quadratic Casimir  $C_2$  in  $R$  by  $C_2(R)$ , then we can rewrite (2.15) as<sup>2</sup>

$$\sum_{a_1, \dots, a_l} \text{Tr} \left( \hat{J}_0^{a_1} \hat{J}_0^{a_1} \dots \hat{J}_0^{a_l} \hat{J}_0^{a_l} q^{L_0} \right) = \sum_n q^n \sum_R m_{n,R} \frac{C_2(R)^l}{k^l} \dim(R). \quad (2.17)$$

The genus  $g$  partition function thus determines these generating series for any  $l \leq g - 1$ . More generally, by choosing a different ordering for the integrals, the genus  $g$  partition function also determines the expressions

$$\sum_{a_1, \dots, a_{2l}} \text{Tr} \left( \hat{J}_0^{a_{\sigma(1)}} \hat{J}_0^{a_{\sigma(2)}} \dots \hat{J}_0^{a_{\sigma(2l-1)}} \hat{J}_0^{a_{\sigma(2l)}} q^{L_0} \right) \prod_{i=1}^l \delta_{a_i a_{i+l}}, \quad (2.18)$$

where  $\sigma$  is any permutation in  $S_{2l}$  (and again  $l \leq g - 1$ ). In analogy to (2.17) the coefficient of  $q^n$  in (2.18) can then be expressed in terms of (in general higher order) Casimir operators.

In the following we shall study the information that can be obtained in this manner systematically. In particular, we shall show (see section 5) that these amplitudes determine the affine Lie algebra that is defined by the currents uniquely. Before we delve into this analysis, it may be instructive to study a few simple cases first.

### 3 Applications and results

It follows from the considerations of the previous section that the genus  $g$  vacuum amplitude determines the expression (2.17). In particular, if the genus  $g$  partition function of two meromorphic conformal field theories agrees, so must the expressions (2.17) for  $l \leq g - 1$ . For many theories the right hand side of (2.17) can be evaluated fairly easily. Thus we may turn the logic around: if (2.17) is different for two conformal field theories for a given  $l$ , then the genus  $g = l + 1$  vacuum amplitude of the two theories must be different. In this section we shall apply these ideas to meromorphic conformal field theories at  $c \leq 32$ .

In all examples we have considered we find that the theories can be distinguished by some higher genus vacuum amplitude. For small values of the central charge (i.e. for

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<sup>2</sup>For simplicity of notation we are assuming here that all simple factors of  $\mathfrak{g}$  have the same level  $k$ ; otherwise we need to rescale the currents of the different simple factors differently.

$c \leq 24$ ), we typically have to go up to genus  $g \geq 5$  in order to distinguish theories; for  $c = 32$ , on the other hand, the discrepancy typically occurs already at genus  $g = 2$ . This behaviour is a consequence of the structure of higher genus modular forms; this will be explained in section 4.

Self-dual meromorphic conformal field theories only exist at central charges that are integer multiples of 8 [26]. The simplest examples are the theories of  $c$  chiral bosons on an even unimodular lattice  $\Lambda$  of rank  $c$ . For such theories, the sub-lattice  $\Lambda_2 \subseteq \Lambda$  generated by its elements of length squared two is the root lattice of some Lie algebra  $\mathfrak{g}$ , and the theory corresponding to  $\Lambda$  contains the affine Kac-Moody algebra  $\hat{\mathfrak{g}}$  at level 1 as a subalgebra. In most cases the theories therefore have an interesting Lie symmetry, and the constraints coming from (2.17) are powerful.

For  $c = 8$  and  $c = 16$ , it is believed that all self-dual conformal field theories are such lattice theories. In fact, for  $c = 8$ , the only self-dual conformal field theory is believed to be the lattice theory based on the  $e_8$  root lattice  $\Gamma_{e_8}$ ; this theory is equivalent to the  $e_8$  level  $k = 1$  affine vertex operator algebra (VOA). The situation is more interesting for  $c = 16$  where two self-dual theories are known (and believed to be the only self-dual theories): the lattice theory based on  $\Gamma_{e_8} \oplus \Gamma_{e_8}$  that is equivalent to the  $e_8 \oplus e_8$  affine VOA at level one and that is often referred to as the  $E_8 \times E_8$  theory. And the lattice theory based on  $\Gamma_{16}$ , whose sublattice  $\Lambda_2$  is the root lattice of  $so(32)$ . The latter VOA contains the  $\mathfrak{g} = so(32)$  affine VOA at level  $k = 1$  as a proper subalgebra. At conformal weight 2 this VOA contains a chiral spinor representation of  $so(32)$ , and thus the Lie group symmetry is  $Spin(32)/\mathbb{Z}_2$  (rather than  $SO(32)$ ).

For  $c \geq 24$ , on the other hand, there are additional self-dual conformal field theories that can be obtained as a  $\mathbb{Z}_2$  orbifold from the lattice theories, see in particular [7] for explicit constructions at  $c = 24$ . However, even at  $c = 24$ , it is not believed that these lattice and orbifold theories already account for all self-dual conformal field theories. In fact Schellekens [8] has conjectured that there are additional self-dual conformal field theories whose genus  $g = 1$  partition function and Lie symmetry he determined. The situation for  $c \geq 32$  is less clear; there is already a gigantic number of lattice theories, and they probably only describe a small subset of all the self-dual theories.

In the following we shall study the behaviour of the higher genus amplitudes for the theories at different values of the central charge in turn.

### 3.1 The two self-dual theories at $c = 16$

As mentioned before, at  $c = 16$  there are two different self-dual conformal field theories, the  $E_8 \times E_8$  theory based on  $\Gamma_{e_8} \oplus \Gamma_{e_8}$ , and the  $Spin(32)/\mathbb{Z}_2$  theory based on  $\Gamma_{16}$ . It is well known that their genus one characters agree; in particular, this implies that the graded dimensions  $\dim \mathcal{H}_h$  of the  $E_8 \times E_8$  theory and the  $Spin(32)/\mathbb{Z}_2$  theory are equal for all values of  $h$ . At  $h = 1$ , the former theory contains the  $248 + 248$  states coming from  $e_8 \oplus e_8$ , while the latter theory contains the 496 states coming from the adjoint representation of  $so(32)$ . With respect to this Lie symmetry we can then decompose also the states at higher

conformal weight. For example, at  $h = 2$ , the  $E_8 \times E_8$  theory contains the states

$$E_8 \times E_8 : \quad \mathcal{H}_2 = \left[ \mathbf{1} \otimes (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}) \right] \oplus \left[ (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}) \otimes \mathbf{1} \right] \oplus \left[ \mathbf{248} \otimes \mathbf{248} \right], \quad (3.1)$$

where we have denoted the different  $e8$  representations by their dimension; in particular,  $\mathbf{248}$  is the adjoint representation, and the Dynkin labels of  $\mathbf{3875} = [1, 0, 0, 0, 0, 0, 0, 0]$ .<sup>3</sup> For later convenience we also give the values of the quadratic Casimirs

$$\begin{aligned} C_2(\mathbf{1} \otimes \mathbf{1}) &= 0, & C_2(\mathbf{1} \otimes \mathbf{248}) &= C_2(\mathbf{248} \otimes \mathbf{1}) = 60 \\ C_2(\mathbf{248} \otimes \mathbf{248}) &= 120 & C_2(\mathbf{1} \otimes \mathbf{3875}) &= C_2(\mathbf{3875} \otimes \mathbf{1}) = 96. \end{aligned} \quad (3.2)$$

For  $\text{Spin}(32)/\mathbb{Z}_2$  the decomposition is

$$\text{Spin}(32)/\mathbb{Z}_2 : \quad \mathcal{H}_2 \equiv \mathbf{1} \oplus \mathbf{496} \oplus \mathbf{527} \oplus \mathbf{35960} \oplus \mathbf{32768}, \quad (3.3)$$

where, in terms of Dynkin labels

$$\begin{aligned} \mathbf{1} &\equiv [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] & C_2(\mathbf{1}) &= 0 \\ \mathbf{496} &\equiv [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] & C_2(\mathbf{496}) &= 60 \\ \mathbf{527} &\equiv [2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] & C_2(\mathbf{527}) &= 64 \\ \mathbf{35960} &\equiv [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] & C_2(\mathbf{35960}) &= 112 \\ \mathbf{32768} &\equiv [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] & C_2(\mathbf{32768}) &= 124, \end{aligned}$$

and we have again given the eigenvalues of the quadratic Casimir in each case. One easily checks that the total dimension of  $\mathcal{H}_2$  is the same in both cases (namely 69752).

It has been known for some time that the vacuum amplitudes of the  $E_8 \times E_8$  and  $\text{Spin}(32)/\mathbb{Z}_2$  theories are the same for  $g \leq 4$ . Recently, it has been proved that the two partition functions are different for  $g = 5$  [6]. We want to give an elementary argument for this, using the techniques we have developed above. From what we have said above, the fact that the partition functions are equal for  $g \leq 4$  must in particular mean that the trace of  $C_2^l$  must agree for  $l = 1, 2, 3$ . On the other hand, if (2.17) was different for  $l = 4$ , this would imply that the genus  $g = 5$  amplitudes differ.

Let us study (2.17) for the first few powers of  $q$ . At  $q^1$ , the states in  $\mathcal{H}_1$  contribute. Both theories have  $k = 1$ , and thus the relevant expressions are

$$\begin{aligned} \text{Spin}(32)/\mathbb{Z}_2 : \quad \text{Tr}_{\mathcal{H}_1}(C_2^l) &= \text{Tr}_{ad}(C_2^l) = \dim(\mathfrak{so}(32)) 2^l h^\vee(\mathfrak{so}(32))^l \\ E_8 \times E_8 : \quad \text{Tr}_{\mathcal{H}_1}(C_2^l) &= 2 \text{Tr}_{ad}(C_2^l) = 2 \dim(e8) 2^l h^\vee(e8)^l. \end{aligned} \quad (3.4)$$

Since  $\dim(\mathfrak{so}(32)) = 496 = 2 \dim(e8)$ , and  $h^\vee(\mathfrak{so}(32)) = 30 = h^\vee(e8)$  it follows that there is no discrepancy for any  $l$ .

The situation is however different at  $q^2$ . Given the values of the quadratic Casimir operators given above, it is straightforward to calculate the trace of  $C_2^l$  on  $\mathcal{H}_2$ . Explicitly,

$$\begin{aligned} \text{Spin}(32)/\mathbb{Z}_2 : \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) &= 1 \cdot 0^l + 496 \cdot 60^l + 527 \cdot 64^l + 35960 \cdot 112^l + 32768 \cdot 124^l \\ E_8 \times E_8 : \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) &= 2 \cdot 0^l + 2 \cdot 248 \cdot 60^l + 2 \cdot 3875 \cdot 96^l + 248 \cdot 248 \cdot 120^l. \end{aligned} \quad (3.5)$$

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<sup>3</sup>We are using the same labelling for the Dynkin labels as LiE.

$\Lambda$	$d24$	$d16 e8$	$(e8)^3$	$a24$	$(d12)^2$	$a17 e7$	$d10 (e7)^2$	$a15 d9$
$N_\Lambda$	1128	744		624	552	456		408
$h_\Lambda^\vee$	46	30		25	22	18		16
$\Lambda$	$(d8)^3$	$(a12)^2$	$a11 d7 e6$	$(e6)^4$	$(a9)^2 d6$	$(d6)^4$	$(a8)^3$	$(a7)^2 (d5)^2$
$N_\Lambda$	360	336	312		264		240	216
$h_\Lambda^\vee$	14	13	12		10		9	8
$\Lambda$	$(a6)^4$	$(a5)^4 d4$	$(d4)^6$	$(a4)^6$	$(a3)^8$	$(a2)^{12}$	$(a1)^{24}$	$u(1)^{24}$
$N_\Lambda$	192	168		144	120	96	72	24
$h_\Lambda^\vee$	7	6		5	4	3	2	-

**Table 1.** Niemeier lattices  $\Lambda$ , number  $N_\Lambda$  of currents and dual Coxeter number  $h_\Lambda^\vee$  of each simple Lie algebra factor.

One then finds that the two expressions agree for  $l = 1, 2, 3$ , but disagree for  $l = 4, 5, \dots$ . In particular, this provides an independent (and elementary) proof that the two partition functions disagree for  $g = 5, 6, \dots$ . Our analysis is also compatible with the known fact that they agree for  $g \leq 4$ .

### 3.2 The self-dual theories at $c = 24$

There are 24 even unimodular lattices (Niemeier lattices) of rank 24, each one corresponding to a distinct meromorphic conformal field theory. The theory based on the Leech lattice, has an abelian Lie algebra symmetry  $u(1)^{24}$ , whereas in all the other cases the Lie algebra is non-abelian and semi-simple.

If two such theories have a different number of currents, the partition function is obviously different already at genus  $g = 1$ . On the other hand, modular invariance of the genus 1 character implies (see section 4) that the genus 1 partition function for the lattice  $\Lambda$  depends only on the number  $N = N_\Lambda$  of currents, i.e. on the number of elements of length squared two in the lattice  $\Lambda$ . Among the 24 Niemeier lattices, there are five pairs of lattices that have the same number  $N_\Lambda$ ; they are listed in table 1 (as customary, Niemeier lattices are denoted by the Lie algebras whose root lattice is generated by the elements of length squared two).

In all cases (except the Leech lattice) the Lie algebra  $\mathfrak{g} = \oplus \mathfrak{g}_i$  is the direct sum of simply laced simple Lie algebras  $\mathfrak{g}_i$ . Furthermore, the dual Coxeter number is the same for all the simple algebras that appear in a given lattice,  $h_\Lambda^\vee = h^\vee(\mathfrak{g}_i)$  for all  $i$ . For any simply laced simple Lie algebra  $\mathfrak{g}$ , the dual Coxeter number  $h^\vee(\mathfrak{g})$  is related to the rank  $r(\mathfrak{g})$  and the dimension  $\dim(\mathfrak{g})$  of  $\mathfrak{g}$  as

$$r(\mathfrak{g}) \left( h^\vee(\mathfrak{g}) + 1 \right) = \dim(\mathfrak{g}) . \quad (3.6)$$

For the Lie algebras  $\mathfrak{g}$  appearing in the Niemeier lattices, the total rank of  $\mathfrak{g}$  is always 24, and hence

$$N_\Lambda \equiv \dim(\mathfrak{g}) = \sum_i \dim(\mathfrak{g}_i) = \sum_i r(\mathfrak{g}_i) (h^\vee(\mathfrak{g}_i) + 1) = (h_\Lambda^\vee + 1) \sum_i r(\mathfrak{g}_i) = 24 (h_\Lambda^\vee + 1) . \quad (3.7)$$

Thus  $h_\Lambda^\vee$  actually only depends on  $N_\Lambda$  as  $h_\Lambda = \frac{N_\Lambda}{24} - 1$ , and hence

$$\text{Tr}_{\mathcal{H}_1}(C_2^l) = \sum_i (2h^\vee(\mathfrak{g}_i))^l \dim(\mathfrak{g}_i) = (2h_\Lambda^\vee)^l N = \left(\frac{N_\Lambda}{12} - 2\right)^l N_\Lambda, \quad (3.8)$$

so that two theories with the same number of currents cannot be distinguished by the trace  $\text{Tr}_{\mathcal{H}_1}(C_2^l)$ , for any  $l$ . As in the case of the  $c = 16$  theories, let us therefore consider the trace of the powers of the quadratic Casimir over  $\mathcal{H}_2$ . The results can be determined from the decomposition of  $\mathcal{H}_2$  in terms of representations of  $\mathfrak{g}$  (see appendix A), and are given in table 2.

It is striking that in all cases  $\text{Tr}_{\mathcal{H}_2}(C_2^l)$  agrees for  $l = 1, 2, 3$ , but disagrees for  $l = 4$ . As in the situation at  $c = 16$  this proves that the partition functions are different for genus  $g = 5$ . It also suggests that the partition functions may be the same for  $g \leq 4$ . We shall prove that this is in fact so in section 4.

It is interesting to apply the same analysis also to theories that are not lattice theories, in particular, to the  $\mathbb{Z}_2$  orbifold theories constructed in [7]. The orbifold theory with affine Kac Moody symmetry  $\widehat{d9}_2 \widehat{a7}_1$  has the same number of currents ( $N = 216$ ) as the lattice theory  $(a7)^2(d5)^2$ , and similarly for the orbifold theory with affine symmetry  $\widehat{d8}_2 (\widehat{b4}_1)^2$  and the lattice theory  $(a6)^4$  ( $N = 192$ ). The explicit results for the trace of  $C_2^l$  over  $\mathcal{H}_2$  are described in table 3 and it shows exactly the same behaviour as for the pairs of Niemeier lattice theories.

The pattern also continues for the theories that were conjectured to exist in [8]. If we include these theories into our considerations, then there are many more cases where the genus  $g = 1$  partition functions agree. For example the theories with affine Lie symmetry  $\widehat{e7}_3 \oplus \widehat{a5}_1$  and  $\widehat{e6}_2 \oplus \widehat{c5}_1 \oplus \widehat{a5}_1$  have the same number of currents ( $N = 168$ ) as the lattice theories  $(a5)^4 d4$  and  $(d4)^6$ . Again, we have compared the trace of  $C_2^l$  in  $\mathcal{H}_2$ , and the results are described in table 4.

Summarising our findings, it appears that we can distinguish self-dual conformal field theories with  $c \leq 24$  by determining their vacuum partition function at genus  $g = 5$ . On the other hand, the genus  $g$  partition functions with  $g \leq 4$  always seem to agree if the two theories in question have the same central charge and the same number of currents (and hence the same torus partition function). In section 4, we shall explain this phenomenon by studying the constraints of modular invariance and factorisation systematically. In fact, we shall be able to show that for  $c \leq 24$  the partition functions at low genera are uniquely determined by the number of currents.

As the central charge increases, such constraints become weaker. In particular, for  $c = 32$ , only the genus 1 partition function is completely determined by the number of currents  $N$ . One may then expect that the discrepancies between partition functions of different theories already occur for lower genera. We have tested this idea by comparing the partition functions for a few pairs of lattice theories that have the same number of currents, and our findings suggest that for  $c = 32$  different theories typically have already different genus  $g = 2$  partition functions (see table 5).<sup>4</sup> At  $c = 32$  the simple algebras  $\mathfrak{g}_i$

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<sup>4</sup>Note however, that for the pair  $(a1)^4(a5)^4 d8$  and  $(a3)^6(d7)^2$  the discrepancy only seems to appear at genus  $g = 3$ . At  $c = 32$  the lattices are not uniquely determined by their Lie algebras any more; in particular, there are more than one theories whose Lie symmetry is  $(d8)^4$ . The entries in table 5 are insensitive to which of these theories one considers, but one can distinguish them using the methods of section 5.

$N_\Lambda = 744$	$d16 e8$	$(e8)^3$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	23302080	23302080	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	2766787200	2766787200	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	329282496000	329282496000	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	39259975772160	39257415936000	2559836160	5
$N_\Lambda = 456$	$a17 e7$	$d10 (e7)^2$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	14544576	14544576	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	1077611904	1077611904	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	80016837120	80016837120	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	5952213614592	5952029755392	183859200	5
$N_\Lambda = 312$	$a11 d7 e6$	$(e6)^4$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	10041408	10041408	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	513437184	513437184	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	26303367168	26303367168	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	1349589196800	1349565235200	23961600	5
$N_\Lambda = 264$	$(a9)^2 d6$	$(d6)^4$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	8521920	8521920	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	369747840	369747840	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	16071221760	16071221760	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	699537653760	699528529920	9123840	5
$N_\Lambda = 168$	$(a5)^4 d4$	$(d4)^6$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	5455296	5455296	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	151466112	151466112	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	4211633664	4211633664	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	117240496128	117239851008	645120	5

**Table 2.** Traces  $\text{Tr}_{\mathcal{H}_2}(C_2^l)$  for CFTs corresponding to Niemeier lattices ( $c = 24$ ). We compare the results between theories with the same number of currents  $N_\Lambda$ .

that appear in  $\mathfrak{g} = \oplus \mathfrak{g}_i$  have different dual Coxeter numbers, and one thus expects that it is already sufficient to compare the traces of  $C_2^l$  over  $\mathcal{H}_1$  (rather than  $\mathcal{H}_2$ ). This is indeed borne out by our analysis (see table 5).

#### 4 Modular properties of partition functions

In the previous section, we compared pairs of meromorphic conformal field theories of the same central charge and with the same number of currents. The general behavior seems

$N = 216$	$(a7)^2(d5)^2$	$\widehat{d9}_2 \widehat{a7}_1$	difference	g
$\dim(H_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	6993216	6993216	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	248949504	248949504	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	8876805120	8876805120	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	316928581632	316924952832	3628800	5
$N = 192$	$(a6)^4$	$\widehat{d8}_2 \widehat{b4}_1^2$	difference	g
$\dim(\mathcal{H}_2)$	196884	196884	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	6225408	6225408	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	197266944	197266944	0	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	6260610048	6260610048	0	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	198933288960	198929660160	3628800	5

**Table 3.** Comparison between  $\mathbb{Z}_2$ -twisting theories of [7] and lattice theories ( $c = 24$ ) with the same number of currents.

$N = 168$	$\widehat{e7}_3 \widehat{a5}_1$	$\widehat{e6}_2 \widehat{c5}_1 \widehat{a5}_1$	$(a5)^4 d4$	$(d4)^6$	g
$\dim(\mathcal{H}_2)$	196884	196884	196884	196884	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	5455296	5455296	5455296	5455296	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	151466112	151466112	151466112	151466112	3
$\text{Tr}_{\mathcal{H}_2}(C_2^3)$	4211633664	4211633664	4211633664	4211633664	4
$\text{Tr}_{\mathcal{H}_2}(C_2^4)$	1172 <b>376</b> 28928	1172 <b>394</b> 59328	1172 <b>404</b> 96128	1172 <b>398</b> 51008	5

**Table 4.** Comparing two of the theories of [8] with lattice theories.

to depend on the central charge: for  $c \leq 24$  the partition functions first differ at genus  $g = 5$ , whereas for  $c = 32$  the difference generically already appears at genus  $g = 2$ . In this section we analyse the consistency conditions of the partition functions, in particular, modular invariance and factorisation properties, systematically. We shall show that for self-dual theories with  $c \leq 24$  the number of currents determines the partition functions for genera  $g \leq 4$  uniquely. On the other hand, for  $c = 32$ , the number of currents only determines the genus  $g = 1$  partition function.

#### 4.1 Generalities

In general, the genus  $g$  partition function of a (not necessarily meromorphic) conformal field theory is not a function on the moduli space  $\mathcal{M}_g$ , but rather a section of the line bundle  $\lambda^{c/2} \otimes \bar{\lambda}^{\bar{c}/2}$ , where  $\lambda$  is the determinant line bundle on  $\mathcal{M}_g$ . In particular, for a meromorphic conformal field theory, the generalized character  $\chi$  is a holomorphic section of the holomorphic line bundle<sup>5</sup>  $\lambda^{c/2}$ .

<sup>5</sup>We observe that  $\lambda^{c/2}$  is a well-defined line bundle on  $\mathcal{M}_g$  only if  $c$  is multiple of 4, which is the case for meromorphic conformal field theories. In the other cases, it can only be defined as a *projective* line bundle [9, 11].



$N = 240$	$(a3)^4 (d5)^4$	$(a3)^8 d8$	difference	g
$\dim(\mathcal{H}_1) = N$	240	240	0	1
$\text{Tr}_{\mathcal{H}_1}(C_2)$	3360	4320	- 960	2
$\text{Tr}_{\mathcal{H}_1}(C_2^2)$	49920	101760	- 51840	3
$\dim(\mathcal{H}_2)$	199024	199024	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	5735040	5258880	476160	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	167260800	149961600	17299200	3
$N = 272$	$(a3)^6 (d7)^2$	$(a1)^4 (a5)^4 d8$	difference	g
$\dim(\mathcal{H}_1) = N$	272	272	0	1
$\text{Tr}_{\mathcal{H}_1}(C_2)$	5088	5088	0	2
$\text{Tr}_{\mathcal{H}_1}(C_2^2)$	110592	114432	- 3840	3
$\dim(\mathcal{H}_2)$	206960	206960	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	6387072	6387072	0	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	205022592	206266752	- 1244160	3
$N = 480$	$(d8)^4$	$(a1)^2 (a9)^2 d12$	difference	g
$\dim(\mathcal{H}_1) = N$	480	480	0	1
$\text{Tr}_{\mathcal{H}_1}(C_2)$	13440	16128	- 2688	2
$\text{Tr}_{\mathcal{H}_1}(C_2^2)$	376320	613632	- 237312	3
$\dim(\mathcal{H}_2)$	258544	258544	0	1
$\text{Tr}_{\mathcal{H}_2}(C_2)$	15048960	13715712	1333248	2
$\text{Tr}_{\mathcal{H}_2}(C_2^2)$	878476800	757969920	120506880	3

**Table 5.** Comparison between some lattice conformal field theories at  $c = 32$  with the same number of currents.

The determinant line bundle  $\lambda$  can be described as follows. Consider the vector bundle  $\Lambda_g$  of rank  $g$  on  $\mathcal{M}_g$ , whose fiber at the point corresponding to the Riemann surface  $\Sigma$  is the  $g$ -dimensional vector space of holomorphic 1-differentials on  $\Sigma$ . As shown in appendix B.1, the choice of a symplectic basis for the first homology group  $H_1(\Sigma, \mathbb{Z})$  determines a basis  $\{\omega_1, \dots, \omega_g\}$  of holomorphic 1-differentials on  $\Sigma$ , and hence a basis of local sections on  $\Lambda_g$ , which we also denote by  $\omega_1, \dots, \omega_g$ . The determinant line bundle  $\lambda$  is then defined as the  $g$ -th exterior product of  $\Lambda_g$ , and given a choice of a basis for  $H_1(\Sigma, \mathbb{Z})$ ,  $\omega_1 \wedge \dots \wedge \omega_g$  defines a local holomorphic section in  $\lambda$ . Under a symplectic transformation (see appendix B.1) the corresponding local section of  $\lambda$  transforms as

$$\omega_1 \wedge \dots \wedge \omega_g \mapsto \det(C\Omega + D)^{-1} (\omega_1 \wedge \dots \wedge \omega_g), \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}). \quad (4.1)$$

The generalised character  $\chi_g$  of a meromorphic CFT is a global holomorphic section of  $\lambda^{c/2}$ , so that it can be written locally as

$$\chi_g = W_g(\Omega) (\omega_1 \wedge \dots \wedge \omega_g)^{c/2},$$

where  $W_g$  is a holomorphic function on the space  $\mathcal{J}_g \subset \mathfrak{H}_g$  of period matrices of Riemann surfaces. Since the section cannot depend on the choice of the local trivialization,  $W_g$  must

transform as a modular form of weight  $c/2$

$$W_g \left( (A\Omega + B)(C\Omega + D)^{-1} \right) = \det(C\Omega + D)^{c/2} W_g(\Omega), \quad (4.2)$$

under the action of  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ . In particular, for lattice theories, the function  $W_g$  is given by

$$W_g^\Lambda(\Omega) = \Theta_\Lambda^{(g)}(\Omega), \quad (4.3)$$

where

$$\Theta_\Lambda^{(g)}(\Omega) = \sum_{\lambda_1, \dots, \lambda_g \in \Lambda} e^{\pi i \sum_{i,j} \Omega_{ij}(\lambda_i, \lambda_j)} \quad (4.4)$$

is the degree  $g$  theta series of  $\Lambda$ .

In section 2.1, we considered the generalised character as a holomorphic function on the Schottky space  $\mathfrak{S}_g$ . As explained in appendix B.3, the space  $\mathfrak{S}_g$  of normalised Schottky groups is a finite covering  $\mathfrak{S}_g \rightarrow \mathcal{M}_g$  of the moduli space. The choice of a Schottky group uniformising the Riemann surface  $\Sigma$  canonically determines a set of  $\alpha$ -cycles and hence a basis  $\omega_1, \dots, \omega_g$  on  $\Sigma$ . This implies that the pull-back of the determinant line bundle  $\lambda_g$  to  $\mathfrak{S}_g$  is isomorphic to the trivial line bundle. Thus, the only ambiguity in the identification of  $\chi_g$  with a holomorphic function on the Schottky space amounts to the choice of a trivialisation. For our purposes we only need the  $g = 1$  result

$$\chi_1 = q^{\frac{c}{24}} (\eta^2)^{-c/2} W_1, \quad (4.5)$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m), \quad q = e^{2\pi i \tau} \quad (4.6)$$

is the Dedekind eta-function. For example, for the conformal field theory corresponding to the unimodular lattice  $\Lambda$ , this formula reproduces the known result

$$\chi_1^\Lambda = q^{\frac{c}{24}} \eta^{-c}(\tau) \Theta_\Lambda^{(g=1)}(\tau). \quad (4.7)$$

Apart from these modular properties, the partition function  $W_g(\Omega)$  must also obey factorisation constraints. Let us consider a family  $\Sigma_t$  of Riemann surfaces of genus  $g$  that, in the limit  $t \rightarrow 0$ , degenerate to a singular surface given by two components of genus  $k$  and  $g - k$  joined by a node. At leading order in the degeneration parameter, the local section  $(\omega_1 \wedge \dots \wedge \omega_g)^{c/2}$  factorises

$$(\omega_1 \wedge \dots \wedge \omega_g)^{c/2} \rightarrow (\omega_1 \wedge \dots \wedge \omega_k)^{c/2} \otimes (\omega_{k+1} \wedge \dots \wedge \omega_g)^{c/2}, \quad (4.8)$$

where  $\omega_1, \dots, \omega_k$  and  $\omega_{k+1}, \dots, \omega_g$  are holomorphic 1-differentials on the components of genus  $k$  and  $g - k$ , respectively. The Riemann period matrix of such a singular surface is simply block-diagonal

$$\lim_{t \rightarrow 0} \Omega_t = \Omega_{k, g-k} \equiv \begin{pmatrix} \Omega^{(k)} & 0 \\ 0 & \Omega^{(g-k)} \end{pmatrix}, \quad (4.9)$$

where  $\Omega^{(k)}$  and  $\Omega^{(g-k)}$  are the period matrices of the two components. The matrix  $\Omega_{k,g-k}$  corresponds to an element of the boundary of the compactification  $\bar{\mathcal{J}}_g$  in  $\mathfrak{H}_g$ . This implies that, in the limit  $\Omega \rightarrow \Omega_{k,g-k}$ , taken along any path in  $\mathcal{J}_g \subseteq \mathfrak{H}_g$ ,  $W_g$  factorises as

$$\lim_{t \rightarrow 0} W_g(\Omega_t) = W_{g-k} \left( \Omega^{(g-k)} \right) W_k \left( \Omega^{(k)} \right) . \quad (4.10)$$

Finally, since the vacuum is unique, we have the normalisation condition

$$\lim_{\tau \rightarrow i\infty} W_1(\tau) = 1 . \quad (4.11)$$

Before we analyse these constraints in more detail, it is useful to introduce some notation. For a general modular form  $f_g$  of degree  $g$  we can always consider the degeneration limit (4.9); in this limit we can always write

$$\lim_{t \rightarrow 0} f_g(\Omega_t) = f_g \begin{pmatrix} \Omega^{(k)} & 0 \\ 0 & \Omega^{(g-k)} \end{pmatrix} = f_k \left( \Omega^{(k)} \right) f_{g-k} \left( \Omega^{(g-k)} \right) , \quad (4.12)$$

where  $f_k$  and  $f_{g-k}$  are modular forms of degree  $k$  and  $g-k$ , respectively. We shall use the symbolic notation

$$f_g \rightarrow f_k \otimes f_{g-k} \quad (4.13)$$

for this factorisation property. It is also useful to introduce the Siegel operator  $\Phi$ , mapping modular forms of degree  $g$  to modular forms of degree  $g-1$ ; it is defined by

$$(\Phi(f_g)) \left( \Omega^{(g-1)} \right) = \lim_{\tau \rightarrow i\infty} f_g \begin{pmatrix} \tau & 0 \\ 0 & \Omega^{(g-1)} \end{pmatrix} . \quad (4.14)$$

The operator  $\Phi$  is linear and is compatible with the product of modular forms

$$\Phi(f_g h_g) = \Phi(f_g) \Phi(h_g) . \quad (4.15)$$

The elements of its kernel, i.e. the modular forms  $f_g$  such that  $\Phi(f_g) = 0$  are called cusp forms of degree  $g$ . Note that if a modular form  $f_g$  of degree  $g$  factorises as  $f_g \rightarrow f_1 \otimes f_{g-1}$  in the limit  $\Omega \rightarrow \Omega_{1,g-1}$ , then

$$\Phi(f_g) = \Phi(f_1) f_{g-1} . \quad (4.16)$$

In particular, using (4.10) and (4.11), it follows that

$$\Phi(W_g) = W_{g-1} \quad (4.17)$$

for each  $g \geq 1$ .

## 4.2 The case of low genera $g \leq 3$

Let us first concentrate on the case where the genus  $g$  satisfies  $g \leq 3$ . (We shall come back to the case of  $g = 4$  below.) In this case the closure of the locus of Riemann period matrices  $\bar{\mathcal{J}}_g$  coincides with the Siegel upper half space  $\mathfrak{H}_g$ , and thus  $W_g$  must be a Siegel modular form (see appendix B.1). The theory of Siegel modular forms is well developed for  $g \leq 3$ , and we can thus be fairly explicit. Let us first review the salient features that will be important for us.

**Genus  $g = 1$ .** At genus  $g = 1$ , the ring of modular forms is generated by the Eisenstein series

$$\phi_4, \quad \phi_6, \quad (4.18)$$

of weight 4 and 6, respectively. We choose the convention that the leading term of both  $\phi_4$  and  $\phi_6$  is 1, i.e. that

$$\Phi(\phi_4) = \Phi(\phi_6) = 1. \quad (4.19)$$

Then the discriminant of the elliptic curve

$$\Delta = \frac{\phi_4^3 - \phi_6^2}{1728} = \eta^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots, \quad q = e^{2\pi i\tau} \quad (4.20)$$

is a cusp form of weight 12 (since its leading coefficient vanishes). In fact,  $\Delta$  generates the ideal of cusp forms at genus  $g = 1$ .

**Genus  $g = 2$ .** The ring of modular forms of degree  $g = 2$  is generated by [27]

$$\psi_4, \quad \psi_6, \quad \chi_{10}, \quad \chi_{12}. \quad (4.21)$$

In our conventions, the Siegel operator satisfies

$$\Phi(\psi_4) = \phi_4, \quad \Phi(\psi_6) = \phi_6, \quad \Phi(\chi_{10}) = \Phi(\chi_{12}) = 0, \quad (4.22)$$

and thus  $\chi_{10}$  and  $\chi_{12}$  are cusp forms. Furthermore, we have the factorisation properties

$$\psi_4 \rightarrow \phi_4 \otimes \phi_4, \quad \psi_6 \rightarrow \phi_6 \otimes \phi_6, \quad \chi_{10} \rightarrow 0, \quad \chi_{12} \rightarrow \Delta \otimes \Delta. \quad (4.23)$$

It is useful to define the modular form  $\psi_{12} = (\psi_4^3 - \psi_6^2)/1728$  of weight 12, which satisfies the properties

$$\Phi(\psi_{12}) = \Delta, \quad \psi_{12} \rightarrow \phi_4^3 \otimes \Delta + \Delta \otimes \phi_4^3 - 1728 \Delta \otimes \Delta, \quad (4.24)$$

as follows from a simple computation.

**Genus  $g = 3$ .** The ring of modular forms is generated by 34 modular forms; the generators with weight not greater than 12 are [28]

$$\alpha_4, \quad \alpha_6, \quad \alpha_{10}, \quad \alpha_{12}, \quad \beta_{12}. \quad (4.25)$$

We choose the conventions that the Siegel operator acts as

$$\Phi(\alpha_4) = \psi_4, \quad \Phi(\alpha_6) = \psi_6, \quad \Phi(\alpha_{10}) = \chi_{10}, \quad \Phi(\alpha_{12}) = \chi_{12}, \quad \Phi(\beta_{12}) = 0, \quad (4.26)$$

and hence  $\beta_{12}$  is a cusp form. Furthermore, in the limit where the genus  $g = 3$  surface degenerates into two surfaces of  $g = 2$  and  $g = 1$ , we have the factorisation properties

$$\alpha_4 \rightarrow \psi_4 \otimes \phi_4, \quad \alpha_6 \rightarrow \psi_6 \otimes \phi_6, \quad \alpha_{10} \rightarrow \chi_{10} \otimes \phi_4 \phi_6, \quad \alpha_{12} \rightarrow \chi_{12} \otimes \phi_4^3 + \psi_{12} \otimes \Delta, \quad (4.27)$$

as well as

$$\beta_{12} \rightarrow \chi_{12} \otimes \Delta. \quad (4.28)$$

We also define the modular form  $\tilde{\alpha}_{12} = (\alpha_4^3 - \alpha_6^2)/1728$ , which satisfies the properties

$$\Phi(\tilde{\alpha}_{12}) = \psi_{12}, \quad \tilde{\alpha}_{12} \rightarrow \psi_{12} \otimes \phi_4^3 + \psi_4^3 \otimes \Delta - 1728 \psi_{12} \otimes \Delta. \quad (4.29)$$

We have now collected all the relevant material to discuss the constraints on  $W_g$  that come from (4.2) together with its factorisation property (4.10). The analysis depends on the value of the central charge, so we need to do the analysis for the different cases separately.

**4.2.1 The case  $c = 8$  and  $c = 16$**

For  $c = 8$ ,  $W_g$  is a modular form of weight 4, while for  $c = 16$  the modular weight of  $W_g$  is 8. For  $g \leq 3$  there is always a unique modular form of weight four and eight, respectively, and hence  $W_g$  must be proportional to that modular form. Using the constraint (4.17) as well as (4.11) it then follows that

$$W_1 = \phi_4^{c/8}, \quad W_2 = \psi_4^{c/8}, \quad W_3 = \alpha_4^{c/8}. \quad (4.30)$$

Since for  $c = 8$  one such theory is the theory based on the  $e8$  lattice, it follows that we must have the identifications

$$\phi_4 = \Theta_{e8}^{(g=1)}, \quad \psi_4 = \Theta_{e8}^{(g=2)}, \quad \alpha_4 = \Theta_{e8}^{(g=3)}, \quad (4.31)$$

where  $\Theta_{e8}$  is the theta series for the  $e8$  lattice. In fact, by (4.5), we can compute the partition function of the  $E_8$  theory,  $\chi_1^{e8}$ , using this approach, and we reobtain the known result

$$\chi_1^{e8} = \frac{q^{1/3}}{\Delta^{1/3}} \phi_4 = q^{1/3} j(\tau)^{1/3} = 1 + 248q + 4124q^2 + \dots, \quad (4.32)$$

where

$$j(\tau) = \frac{\phi_4^3}{\Delta} = \frac{1}{q} + 744 + 196884q + \dots \quad (4.33)$$

is the  $j$ -invariant.

For  $c = 16$ , on the other hand, there are two self-dual theories, namely the  $E_8 \times E_8$  and the  $\text{Spin}(32)/\mathbb{Z}_2$  theories. The above argument implies that both must have the same partition function for  $g = 1, 2, 3$ , namely the one given by (4.30). This obviously ties in with our findings of section 3.1.

**4.2.2 The case  $c = 24$**

The case  $c = 24$  is actually the most interesting one from this point of view. At  $c = 24$  we are looking for modular forms of weight 12. At genus one (degree one), the space of modular forms is 2-dimensional and we can take  $\phi_4^3 = \Theta_{e8}^3$  and  $\Delta$  as generators. The condition  $\Phi(W_1) = 1$  implies then

$$W_1 = \phi_4^3 + a\Delta, \quad (4.34)$$

where  $a$  is some constant (that will depend on the theory). The corresponding partition function  $\chi_1$  then is

$$\chi_1 = \frac{q}{\Delta} (\phi_4^3 + a\Delta) = q(j(\tau) + a) = 1 + (744 + a)q + 196884q^2 + \dots. \quad (4.35)$$

The coefficient of  $q$  in this expansion is the number  $N$  of currents of the theory, so that the genus 1 partition function depends only on  $N$

$$W_1 = W_1(N) = \phi_4^3 + (N - 744)\Delta, \quad \chi_1 = q(j + N - 744). \quad (4.36)$$

Let us consider the genus 2 partition function. At grade  $g = 2$  the space of modular forms of weight 12 is 3-dimensional, and it is convenient to write  $W_2$  as a linear combination of  $\psi_4^3$ ,  $\chi_{12}$  and  $\psi_{12}$ . The condition  $\Phi(W_2) = W_1$  now implies that

$$W_2 = \psi_4^3 + (N - 744)\psi_{12} + b\chi_{12}, \quad (4.37)$$

for some constant  $b$ . Next we impose the factorisation condition  $W_2 \rightarrow W_1 \otimes W_1$ . Since

$$W_1 \otimes W_1 = \phi_4^3 \otimes \phi_4^3 + (N - 744)(\phi_4^3 \otimes \Delta + \Delta \otimes \phi_4^3) + (N - 744)^2 \Delta \otimes \Delta \quad (4.38)$$

and since

$$W_2 \rightarrow \phi_4^3 \otimes \phi_4^3 + (N - 744)(\phi_4^3 \otimes \Delta + \Delta \otimes \phi_4^3 - 1728\Delta \otimes \Delta) + b\Delta \otimes \Delta, \quad (4.39)$$

we obtain  $b = (N - 744)(N + 984)$ . Thus we find that

$$W_2 = W_2(N) = \psi_4^3 + (N - 744)\psi_{12} + (N - 744)(N + 984)\chi_{12}, \quad (4.40)$$

and thus also the genus 2 partition function is completely determined by the number of currents. A similar result has also been recently obtained in [14, 15], using a different approach.

The computation at genus 3 is analogous. The modular form  $W_3$  is a linear combination of  $\alpha_4^3$ ,  $\alpha_{12}$ ,  $\tilde{\alpha}_{12}$  and  $\beta_{12}$ , and the constraints are  $\Phi(W_3) = W_2$  and  $W_3 \rightarrow W_2 \otimes W_1$ . The first condition gives

$$W_3 = \alpha_4^3 + (N - 744)\tilde{\alpha}_{12} + (N - 744)(N + 984)\alpha_{12} + d\beta_{12}, \quad (4.41)$$

whereas the second one fixes  $d = (N + 984)(N - 744)^2$ , so that

$$W_3 = W_3(N) = \alpha_4^3 + (N - 744)\left[\tilde{\alpha}_{12} + (N + 984)(\alpha_{12} + (N - 744)\beta_{12})\right]. \quad (4.42)$$

This proves our claim that the partition functions for  $g \leq 3$  at  $c = 24$  are uniquely determined in terms of the number of currents.

It is amusing to observe that the partition function  $\chi_g$ , for genus  $g = 1, 2, 3$ , has a polynomial dependence on the number of currents  $N$ , with the degree of the polynomial being  $g$ . Following our general discussion, this therefore implies that the expressions  $\text{Tr}_{\mathcal{H}_2}(C_2^l)$ , for  $l = 0, 1, 2$ , must have an analogous polynomial dependence on  $N$ , with degree (at most)  $l + 1$ . This holds trivially for the case of  $l = 0$ , since the dimension of  $\mathcal{H}_2$  does not depend on  $N$ , as the explicit expression for  $\chi_1$  shows. For  $l = 1$  and  $l = 2$ , however, this is a non-trivial claim. By considering a few different theories, one can determine the coefficients of the polynomials explicitly, and one finds

$$\text{Tr}_{\mathcal{H}_2}(C_2) = -2N^2 + 32808N, \quad (4.43)$$

$$\text{Tr}_{\mathcal{H}_2}(C_2^2) = -\frac{23N^3}{36} + \frac{16421N^2}{3} + 40N. \quad (4.44)$$

One can then check that these identities are in fact satisfied by *all* meromorphic conformal field theories with central charge  $c = 24$ . This provides a highly non-trivial cross-check of the correctness of the analysis in this section and of the results of section 3.

**4.2.3 The case  $c = 32$**

For theories with central charge  $c = 32$ , the space of modular forms of grade 1 is still 2-dimensional, and we may take the generators to be  $\phi_4^4$  and  $\Delta\phi_4$ . It follows that, in this case,

$$W_1 = \phi_4(\phi_4^3 + (N - 992)\Delta), \tag{4.45}$$

and the genus 1 partition function still depends only on  $N$

$$\chi_1 = q(j(\tau) + (N - 992)) \chi_1(E_8) = 1 + Nq + (248N + 139504)q^2 + \dots \tag{4.46}$$

At genus 2, the space of modular forms is generated by  $\chi_{12}\psi_4$ ,  $\chi_{10}\psi_6$ ,  $\psi_6^2\psi_4$ , and  $\psi_4^4$ . Since  $\chi_{10}$  is a cusp form and vanishes when the period matrix is block diagonal, the coefficient of  $\chi_{10}\psi_6$  is not determined by factorisation constraints. This implies that, in general, a pair of conformal field theories of central charge 32, with the same partition function at  $g = 1$ , may have a different partition function at genus 2. This is very nicely consistent with the explicit computations of section 3.

**4.3 Comments about genus  $g \geq 4$**

The above analysis cannot be generalised to genus  $g > 3$  in a straightforward manner. First of all, for  $g \geq 4$ , the closure  $\bar{\mathcal{J}}_g$  of the locus of Riemann period matrices does not correspond to the whole Siegel upper half-space  $\mathfrak{H}_g$  any longer. This implies that  $W_g$  does not necessarily extend to a well-defined Siegel modular form on  $\mathfrak{H}_g$ . The second issue is that the complete classification of Siegel modular forms of degree  $g > 3$  is not known. For these reasons, a general treatment is not possible for genera  $g > 3$ . However, some results can be obtained for the genus  $g = 4$  partition functions of lattice theories with central charge  $c \leq 24$ .

For  $c = 16$  and  $g = 4$ , the theta series  $\Theta_{e_8}^2$  and  $\Theta_{d16}$  are distinct modular forms on  $\mathfrak{H}_g$ , but their difference vanishes on  $\bar{\mathcal{J}}_4$ . Remarkably,

$$J_8 := \Theta_{d16} - \Theta_{e_8}^2 = 0, \tag{4.47}$$

is in fact the defining equation for  $\bar{\mathcal{J}}_4$  in  $\mathfrak{H}_4$ , thus providing the explicit solution for the Schottky problem at  $g = 4$  [29]. In particular, any modular form vanishing on  $\bar{\mathcal{J}}_4$  must be the product of a modular form times some power of  $J_8$ .

For lattice theories at  $c = 24$ ,  $W_4$  must lie in the subspace of modular forms of degree 4 generated by theta series. Because of (4.17) the image of  $W_4$  under the Siegel operator must be given by  $W_3(N)$  of eq. (4.42), where  $N$  takes all the possible values in table 1. It is easy to see from this expression that the space generated by the different  $W_3(N)$  (where  $N$  attains all the different allowed values) is actually 4-dimensional. In particular, this shows that the whole space of modular forms of degree 3 and weight 12 is generated by theta series. This is true also for modular forms of degree 4 and weight 12 [30]. Furthermore, it is known that the space of cusp forms of degree 4 and weight 12 is two dimensional [31]. One such cusp form is  $\Theta_{e_8}J_8$ , because

$$\Phi(\Theta_{e_8}J_8) = \Phi\left(\Theta_{e_8 d16}^{(4)} - \Theta_{e_8^3}^{(4)}\right) = \Theta_{e_8 d16}^{(3)} - \Theta_{e_8^3}^{(3)} = 0. \tag{4.48}$$

It then follows that the space of modular forms of degree 4 and weight 12 is 6-dimensional, and we can choose a basis to consist of  $\Theta_{E_8} J_8$ ,  $K$ ,  $\xi_4$ ,  $\xi_{12}$ ,  $\tilde{\xi}_{12}$  and  $\rho_{12}$ , where  $K$  is a cusp form and

$$\Phi(\xi_4) = \alpha_4^3, \quad \Phi(\xi_{12}) = \alpha_{12}, \quad \Phi(\tilde{\xi}_{12}) = \tilde{\alpha}_{12}, \quad \Phi(\rho_{12}) = \beta_{12}. \quad (4.49)$$

Then, the theta series of degree 4 can be written as

$$\Theta_{\Lambda}^{(4)} = c_4(N)\xi_4 + c_{12}(N)\xi_{12} + \tilde{c}_{12}(N)\tilde{\xi}_{12} + d_{12}(N)\rho_{12} + e\Theta_{E_8} J_8 + fK,$$

for some coefficients  $c_4(N)$ ,  $c_{12}(N)$ ,  $\tilde{c}_{12}(N)$ ,  $d_{12}(N)$ ,  $e$  and  $f$ , where  $e$  and  $f$  in principle depend on  $\Lambda$ . In fact, the  $c_4(N)$ ,  $c_{12}(N)$ ,  $\tilde{c}_{12}(N)$ ,  $d_{12}(N)$  are uniquely fixed by the condition that

$$\Phi(\Theta_{\Lambda}^{(4)}) = \Theta_{\Lambda}^{(3)} = W_3, \quad (4.50)$$

i.e. they simply agree with the coefficients of  $\alpha_4^3$ ,  $\alpha_{12}$ ,  $\tilde{\alpha}_{12}$  and  $\beta_{12}$  in  $W_3(N)$ . Note that all these coefficients are polynomials of degree at most 3 in  $N$ . In the limit  $\Omega \rightarrow \Omega_{k,4-k}$ ,  $k = 1, 2$ , the theta series satisfy the factorisation conditions

$$\Theta_{\Lambda}^{(4)} \rightarrow \Theta_{\Lambda}^{(k)} \otimes \Theta_{\Lambda}^{(4-k)}, \quad k = 1, 2. \quad (4.51)$$

It is easy to see that, for both  $k = 1$  and  $k = 2$

$$\Theta_{E_8} J_8 \equiv \Theta_{e_8 d_{16}}^{(4)} - \Theta_{(e_8)^3}^{(4)} \rightarrow \Theta_{e_8 d_{16}}^{(k)} \otimes \Theta_{e_8 d_{16}}^{(g-k)} - \Theta_{(e_8)^3}^{(k)} \otimes \Theta_{(e_8)^3}^{(g-k)} = 0. \quad (4.52)$$

We now want to argue that the corresponding factorisation limit of  $K$  cannot be trivial. To see this we note that  $\Theta_{\Lambda}^{(k)} = W_k(N)$  for  $k = 1, 2, 3$ , is a polynomial of degree  $k$  in  $N$ . Thus  $\Theta_{\Lambda}^{(k)} \otimes \Theta_{\Lambda}^{(4-k)}$  is a polynomial of degree 4. On the other hand, the coefficients  $c_4(N)$ ,  $c_{12}(N)$ ,  $\tilde{c}_{12}(N)$ ,  $d_{12}(N)$  are all polynomials of degree at most 3. If the factorisation limit of  $K$  was trivial, the factorisation constraint would lead to an identity between a polynomial of degree at most 3, and a polynomial of degree 4. However, such an identity can at most be true for five different values of  $N$ . But there are 19 possible values for  $N$  in table 1, and it is thus impossible that the identity is true for all of them. It therefore follows that the factorisation limit of  $K$  is non-trivial.

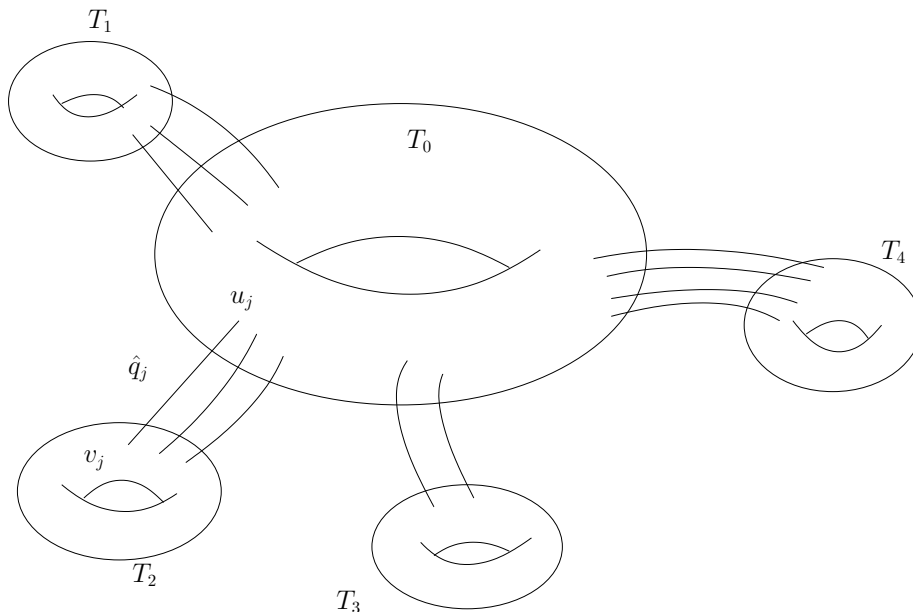
But if the factorisation limit of  $K$  is non-trivial, then we can determine the coefficient of  $K$  via factorisation. By the same argument as above, the coefficient of  $K$  is then a polynomial in  $N$  of degree 4. But since  $\Theta_{e_8} J_8$  vanishes on  $\tilde{\mathcal{J}}$ , this proves that the restrictions of the theta series of degree  $g = 4$  to  $\tilde{\mathcal{J}}_4$  depends only on the number of currents, and that the dependency is polynomial of degree 4.

As in the lower genus case, such an analysis implies that the traces  $\text{Tr}_{\mathcal{H}_2}(C_2^3)$  must be polynomial of degree 4 in the number of currents  $N$ . Again, we can fix the precise coefficients by comparison with a few explicit examples, and we find that

$$\text{Tr}_{\mathcal{H}_2}(C_2^3) = -\frac{133N^4}{864} + \frac{10969N^3}{12} + 2N^2 - 272N. \quad (4.53)$$

It is then again a non-trivial consistency check that the identity also holds for the other Niemeier lattice theories at  $c = 24$ . In fact, the identity actually holds for all known  $c = 24$  theories; this suggests that the above results may be more generally correct.





**Figure 4.** A singular Riemann surface of genus 12, corresponding to  $r = 4$ ,  $l_1 = 3$ ,  $l_2 = 3$ ,  $l_3 = 2$ ,  $l_4 = 4$ . Each line represents a node connecting a torus  $T_i$  to the torus  $T_0$ .

The same argument does not work at genus  $g = 5$ , since at  $g = 5$  there exists a Siegel modular form  $M$  of weight 12 that does not vanish on the moduli space of Riemann surfaces, but for which  $\Phi(M) = \Theta_{e8} J_8$ . The coefficient of  $M$  thus cannot be determined by factorisation arguments, and will therefore depend on the actual structure of the theory. This is obviously in perfect agreement with what we saw explicitly in our analysis of section 3.

## 5 A general approach

The analysis of the previous section suggests that one should be able to identify the Lie symmetry of a given conformal field theory from its genus  $g$  vacuum amplitudes. We now want to show that this is indeed so. A convenient method to approach this problem is to consider more general degeneration limits of genus  $g$  surfaces.

### 5.1 Invariants from partition functions

Given a genus  $g$  Riemann surface we want to consider the degeneration limit that is sketched in figure 4. Its connected components, once the nodes are removed, are  $r + 1$  tori  $T_0, T_1, \dots, T_r$ , with modular parameters  $q_0, q_1, \dots, q_r$ . The torus  $T_j$  is connected by  $l_j$  nodes to the torus  $T_0$ , but there are no nodes connecting two tori  $T_j$  and  $T_k$  with  $k, j > 0$ , and no nodes identifying distinct points on the same torus. Thus, the total number of nodes is  $n = \sum_j^r l_j$ , and each of them is associated with a degeneration parameter  $\hat{q}_j$  and two points  $u_j, v_j$ ,  $j = 1, \dots, n$ , with  $u_j \in T_0$  and  $v_j$  on a torus  $T_i$  for some  $i > 0$ . The genus  $g$  of such a singular Riemann surface can be read off directly from the geometrical sketch: each torus  $T_j$  with the  $l_j$  connecting nodes adds  $l_j$  handles; together with the torus

$T_0$  in the middle, the total genus is therefore  $g = n + 1$ . This also ties in with the counting of the moduli: there are 3 parameters  $\hat{q}_j, u_j, v_j$  associated to each node, and each torus has one modular parameter and one symmetry, so that the total number of independent parameters is  $3n$ . A surface of genus  $g > 1$  has  $3g - 3$  moduli, so that this also gives  $g = n + 1$ .

Let us consider the expansion of the genus  $g = n + 1$  character of a meromorphic CFT in the limit  $\hat{q}_1, \dots, \hat{q}_n \rightarrow 0$ . The coefficient of the term  $\prod_j \hat{q}_j$  is given by a product of  $r$  correlation functions of currents, one per torus,

$$\chi_g |_{\prod_{j=1}^n \hat{q}_j} = \text{Tr}_{\mathcal{H}} \left( q_0^{L_0} \prod_{i=1}^n \hat{J}^{a_i}(u_i) \right) \text{Tr}_{\mathcal{H}} \left( q_1^{L_0} \prod_{i=1}^{l_1} \hat{J}^{a_i}(v_i) \right) \cdots \text{Tr}_{\mathcal{H}} \left( q_r^{L_0} \prod_i \hat{J}^{a_i}(v_i) \right). \tag{5.1}$$

The indices of the  $l_j$  currents appearing in the correlator on the torus  $T_j, j > 0$ , are contracted with a set of  $l_j$  currents in the correlator on  $T_0$ . By integrating all the points  $u_j, v_j$  around the  $\alpha$ -cycles of the respective tori, we pick up the zero modes of the currents and obtain a product of traces

$$\text{Tr}_{\mathcal{H}}(q_0^{L_0} \hat{J}_0^{a_1} \cdots \hat{J}_0^{a_n}) \text{Tr}_{\mathcal{H}}(q_1^{L_0} \hat{J}_0^{a_1} \cdots \hat{J}_0^{a_{l_1}}) \cdots \text{Tr}_{\mathcal{H}}(q_r^{L_0} \hat{J}_0^{a_{n-l_r+1}} \cdots \hat{J}_0^{a_n}). \tag{5.2}$$

Here we have picked some particular order for the integration paths of the points; this is not the most general case (and indeed we could consider more complicated degenerations, for example when there are also nodes between  $T_i$  and  $T_j$  with  $i, j > 0$ ), but for our present purposes, this will suffice.

The product of traces in (5.2) can be expanded in powers of  $q_0, \dots, q_r$ , and the coefficient of the term  $q_0^h q_1^{h_1} \cdots q_r^{h_r}$  is

$$\text{Tr}_{\mathcal{H}_h}(\hat{t}^{a_1} \cdots \hat{t}^{a_n}) \text{Tr}_{\mathcal{H}_{h_1}}(\hat{t}^{a_1} \cdots \hat{t}^{a_{l_1}}) \cdots \text{Tr}_{\mathcal{H}_{h_r}}(\hat{t}^{a_{n-l_r+1}} \cdots \hat{t}^{a_n}), \tag{5.3}$$

where we denote by  $\hat{t}^a$  the rescaled Lie algebra generators (compare (2.9))

$$\hat{t}^a = k^{-\frac{1}{2}} t^a \quad \text{with} \quad t^a \equiv J_0^a, \tag{5.4}$$

and  $k$  is the level of the corresponding Lie algebra. For the following it is convenient to define the Casimir operators of degree  $l$  (see for example [32])

$$C_l^{(\mathfrak{g})} := \text{Tr}_{ad}(t^{a_1} \cdots t^{a_l}) t^{a_1} \cdots t^{a_l}, \quad l = 2, 3, \dots, \tag{5.5}$$

where we sum over an orthonormal basis with respect to the Killing form (see (2.8)), and  $ad$  denotes the adjoint representation of  $\mathfrak{g}$ . For example, for  $l = 2$ , this is just the rescaled quadratic Casimir operator

$$C_2^{(\mathfrak{g})} = 2 h^\vee C_2, \quad \text{since} \quad \text{Tr}_{ad}(t^a t^b) = 2 h^\vee \delta^{ab}. \tag{5.6}$$

In terms of these Casimir operators we can then express (5.3) for  $h_1 = \cdots = h_r = 1$ , with  $h$  being arbitrary, as

$$k^{-n} \text{Tr}_{\mathcal{H}_h} \left( C_{l_1}^{(\mathfrak{g})} C_{l_2}^{(\mathfrak{g})} \cdots C_{l_r}^{(\mathfrak{g})} \right). \tag{5.7}$$

Here we have assumed that the Lie algebra  $\mathfrak{g}$  is simple, so that there is only one level  $k$ ; in general, if  $\mathfrak{g} = \oplus \mathfrak{g}_i$ , where  $\mathfrak{g}_i$  has level  $k_i$ , we get instead of (5.7)

$$\mathrm{Tr}_{\mathcal{H}_h} \left[ \prod_{j=1}^r \left( \sum_i k_i^{-l_j} C_{l_j}^{(\mathfrak{g}_i)} \right) \right]. \quad (5.8)$$

Note that the trace  $\mathrm{Tr}_{\mathcal{H}_1}(t^{a_p} \dots t^{a_{p+l_j}})$  is only non-zero if all generators  $t^{a_s}$ ,  $s = p, \dots, p+l_j$  lie in the same simple Lie algebra  $\mathfrak{g}_i$ .

## 5.2 Identifying the Lie algebra

In the following we want to show that one can determine the affine Lie algebra from suitable degeneration limits of higher genus partition functions. The Lie algebra generators appear at  $h = 1$ , and thus we should consider (5.3) (or (5.7) and (5.8)) for  $h = 1$ . Let us denote the rescaled value of the Casimir operator  $C_l^{(\mathfrak{g})}$  in the adjoint representation  $ad(\mathfrak{g})$  by

$$\xi_l(\mathfrak{g}, k) = \frac{C_l^{(\mathfrak{g})}(ad(\mathfrak{g}))}{k^l}. \quad (5.9)$$

If the affine algebra is a direct sum of simple affine Lie algebras (and  $\hat{u}(1)$  factors),  $\hat{\mathfrak{g}} = \oplus n_i \hat{\mathfrak{g}}_i$ , where  $\hat{\mathfrak{g}}_i$  has level  $k_i$  and the  $n_i$  are multiplicities, then (5.8) becomes simply

$$\sum_i k_i^{-n} \mathrm{Tr}_{\mathcal{H}_1} \left( C_{l_1}^{(\mathfrak{g}_i)} C_{l_2}^{(\mathfrak{g}_i)} \dots C_{l_r}^{(\mathfrak{g}_i)} \right) = \sum_i n_i \dim(\mathfrak{g}_i) \prod_{j=1}^r \xi_{l_j}(\mathfrak{g}_i, k_i). \quad (5.10)$$

By taking linear combinations of such invariants we can obtain any polynomial of the  $\xi_l(\mathfrak{g}_i, k_i)$ , i.e. we can get expressions for

$$\sum_i n_i \dim(\mathfrak{g}_i) P(\xi_2(\mathfrak{g}_i, k_i), \xi_3(\mathfrak{g}_i, k_i), \dots), \quad (5.11)$$

where  $P$  is an arbitrary polynomial. In fact, the vacuum amplitudes up to genus  $g$  gives us access to all polynomials whose total degree is  $g - 1$  (where we regard  $\xi_l(\mathfrak{g}_i, k_i)$  as having degree  $l$ ).

The main strategy for our argument is now as follows. Since the dimension of  $\mathcal{H}_1$  is finite, it is clear that only finitely many possible  $\mathfrak{g}_i$  may appear in  $\mathfrak{g}$ . We can also show (see section 5.2.1 below for the detailed argument) that only finitely many values of  $k_i$  are possible. Thus there are only finitely many possibilities for  $\hat{\mathfrak{g}}_i$  we have to distinguish.

The second ingredient is that any simple affine algebra  $\hat{\mathfrak{g}}$  at level  $k$  is uniquely identified by its values for  $\xi_l$ . More specifically, as shown in more detail below in section 5.2.2, for any pair of simple affine Lie algebras  $\mathfrak{g}_i$  at level  $k_i$  and  $\mathfrak{g}_j$  at level  $k_j$ , for which either  $\mathfrak{g}_i \neq \mathfrak{g}_j$  or  $k_i \neq k_j$ , there exists an  $2 \leq l_{ij} < \infty$  such that

$$\xi_{l_{ij}}(\mathfrak{g}_i, k_i) \neq \xi_{l_{ij}}(\mathfrak{g}_j, k_j). \quad (5.12)$$

Then we can consider the polynomial

$$P_i(x_2, x_3, \dots) = \prod_{j \neq i} \frac{x_{l_{ij}} - \xi_{l_{ij}}(\mathfrak{g}_j, k_j)}{\xi_{l_{ij}}(\mathfrak{g}_i, k_i) - \xi_{l_{ij}}(\mathfrak{g}_j, k_j)}, \quad (5.13)$$

where  $j$  runs over all the finitely many possibilities for  $\hat{\mathfrak{g}}_j$ . If we apply (5.11) with  $P = P_i$ , then we simply obtain  $n_i \dim(\mathfrak{g}_i)$ . This allows us to read off the multiplicity with which  $\hat{\mathfrak{g}}_i$  appears in  $\hat{\mathfrak{g}}$ .

Since (5.11) with  $P = P_i$  can be obtained from a suitable degeneration limit of the vacuum genus  $g$  amplitudes (where  $g$  is sufficiently large such that the degree of all  $P_i$  is less than  $g - 1$ ), this argument allows us to identify  $\hat{\mathfrak{g}}$  uniquely. Put differently, if two meromorphic conformal field theories contain different affine algebras, then their vacuum amplitudes cannot agree for all genera.

As an example, let us consider the  $E_8 \times E_8$  and  $\text{Spin}(32)/\mathbb{Z}_2$  theories. The dual Coxeter numbers are the same, so that  $C_2^{(\mathfrak{g})}(ad(\mathfrak{g}))$  is the same for both theories. However, the two Lie algebras have also a fourth order Casimir  $C_4^{(\mathfrak{g})}$ , which can be obtained from a genus 5 partition function. In the adjoint representations it equals (the details of this computations are explained in section 5.2.2)

$$\begin{aligned} E_8 \times E_8 : \text{Tr}_{\mathcal{H}_1}(t^{a_1} \dots t^{a_4}) \text{Tr}_{\mathcal{H}_1}(t^{a_1} \dots t^{a_4}) &= 2 \dim(e8) C_4^{(\mathfrak{g})}(ad(\mathfrak{g})) = 589248000 \\ \text{Spin}(32)/\mathbb{Z}_2 : \text{Tr}_{\mathcal{H}_1}(t^{a_1} \dots t^{a_4}) \text{Tr}_{\mathcal{H}_1}(t^{a_1} \dots t^{a_4}) &= \dim(d16) C_4^{(\mathfrak{g})}(ad(\mathfrak{g})) = 749237760, \end{aligned} \tag{5.14}$$

and hence allows one to distinguish the two theories at genus  $g = 5$ , in agreement with the earlier analysis.

In order to complete our argument it remains to explain the two remaining issues, namely (i) that there are only finitely many possible affine algebras that may appear; and (ii) that (5.12) holds. We shall first deal with (i).

### 5.2.1 The bound on the level

Since

$$\dim(\mathcal{H}_1) = \sum_i n_i \dim(\mathfrak{g}_i) \tag{5.15}$$

it is clear that only those Lie algebras  $\mathfrak{g}_i$  may appear in  $\mathfrak{g}$  that satisfy  $\dim(\mathfrak{g}_i) \leq \dim(\mathcal{H}_1)$ . Given  $\dim(\mathcal{H}_1)$ , there are therefore only finitely many possibilities for  $\mathfrak{g}_i$ . However, this dimensional reasoning does not give a constraint on the possible levels  $k_i$ . In this section, we will show that the levels are also bounded.

The starting point of our analysis is the quantity

$$A := \text{Tr}_{\mathcal{H}_1}(\hat{t}^a \hat{t}^b) \text{Tr}_{\mathcal{H}_1}(\hat{t}^a \hat{t}^b) = \sum_i \frac{\dim(\mathfrak{g}_i) C_2^{(\mathfrak{g}_i)}(ad(\mathfrak{g}_i))}{k_i^2}, \tag{5.16}$$

that may be obtained from the degeneration of the genus  $g = 2$  vacuum amplitude. By virtue of (5.16)  $A$  is a rational number. We can thus find a positive integer  $M$  such that  $AM \in \mathbb{N}$ , as well as

$$x_i := M \dim(\mathfrak{g}_i) C_2^{(\mathfrak{g}_i)}(ad(\mathfrak{g}_i)) \in \mathbb{N} \tag{5.17}$$

for all  $i$  with  $\dim(\mathfrak{g}_i) \leq \dim(\mathcal{H}_1)$ . By multiplying both sides of (5.16) by  $M$  we then obtain

$$\sum_i \frac{x_i}{k_i^2} = MA \in \mathbb{N}. \tag{5.18}$$

Note that the numerators  $x_i$  are uniformly bounded

$$x_i \leq X, \tag{5.19}$$

for some  $X$ , because each  $x_i$  only depends on the Lie algebra  $\mathfrak{g}_i$  as well as the choice of  $M$ .

Let  $k_1$  be the smallest level that appears in  $\hat{\mathfrak{g}} = \oplus n_i \hat{\mathfrak{g}}_i$ . The right hand side of (5.18) is a positive integer, and hence must at least be equal to 1. On the other hand, the left hand side is a sum over at most  $N = \dim(\mathcal{H}_1)$  positive terms, each of which is bounded by

$$\frac{x_i}{k_i^2} \leq \frac{X}{k_1^2}. \tag{5.20}$$

It therefore follows that

$$N \frac{X}{k_1^2} \geq 1, \tag{5.21}$$

and hence  $k_1$  is bounded by

$$k_1^2 \leq XN. \tag{5.22}$$

If  $k_1$  is the only level appearing in the decomposition of  $\hat{\mathfrak{g}}$ , we are done. Otherwise let us multiply both sides of eq. (5.18) by  $k_1^2$  to obtain

$$\sum_{i \geq 2} \frac{k_1^2 x_i}{k_i^2} = k_1^2 MA - x_1 \in \mathbb{N}. \tag{5.23}$$

We choose our numbering such that  $k_2$  is the second smallest level. Then we repeat the argument where now the numerators  $k_1^2 x_i$  are uniformly bounded by  $X^2 N$ . Since the right hand side is still positive, we thus obtain the inequality

$$(N - 1) \frac{X^2 N}{k_2^2} \geq 1 \quad \Rightarrow \quad k_2^2 \leq X^2 N(N - 1). \tag{5.24}$$

Repeating this procedure (at most  $N$  times) we obtain an upper bound for all possible levels  $k_i$  appearing in the decomposition of  $\hat{\mathfrak{g}}$ .

### 5.2.2 Higher degree Casimir invariants in the adjoint representation

Thus it only remains to prove (5.12) for any pair of affine Lie algebras  $\mathfrak{g}$  at level  $k$  and  $\mathfrak{g}'$  at level  $k'$  for which either  $\mathfrak{g} \neq \mathfrak{g}'$  or  $k \neq k'$ . Given a simple Lie algebra  $\mathfrak{g}$ , consider the linear operator  $Q$  acting on the tensor product representation  $ad \otimes ad$  [33]

$$Q = \sum_a t^a \otimes t^a, \tag{5.25}$$

where  $t^a$  acts in the standard way on  $ad$ . The trace of its  $l$ 'th power equals

$$\text{Tr}_{ad \otimes ad}(Q^l) = \sum_{a_1, \dots, a_l} \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) = \dim(\mathfrak{g}) C_l^{(\mathfrak{g})}(ad(\mathfrak{g})). \tag{5.26}$$

The Lie algebra generators  $t^a$  act on the tensor product  $ad \otimes ad$  as

$$y^a = t^a \otimes \mathbf{1} + \mathbf{1} \otimes t^a. \tag{5.27}$$

Algebra	Eigenvalues					
$u(1)$	0					
$a1$	-4	-2	2			
$a2$	-6	-3	0	2		
$a(r), r > 2$	$-2(r+1)$	$-(r+1)$	-2	0	2	
$b3$	-10	-5	-4	-3	0	2
$b(r), r > 3$	$-2(2r-1)$	$-2r+1$	$-2r+3$	-4	0	2
$c(r), r \geq 2$	$-2(r+1)$	$-(r+2)$	$-(r+1)$	-1	0	2
$d4$	-12	-6	-4	0	2	
$d(r), r > 4$	$-4(r-1)$	$-2r+2$	$-2r+4$	-4	0	2
$e6$	-24	-12	-6	0	2	
$e7$	-36	-18	-8	0	2	
$e8$	-60	-30	-12	0	2	
$f4$	-18	-9	-5	0	2	
$g2$	-8	-4	$-\frac{10}{3}$	0	2	

**Table 6.** The different eigenvalues  $\lambda_i$  (5.30) of  $Q$  for all the simple Lie algebras and for  $u(1)$ . In each row the eigenvalues are given in increasing order.

In terms of these generators we can write the operator  $Q$  as

$$Q = \frac{1}{2} \sum_a \left( y^a y^a - (t^a t^a \otimes \mathbf{1}) - (\mathbf{1} \otimes t^a t^a) \right). \tag{5.28}$$

Let  $ad \otimes ad = \oplus_i R_i$  be the decomposition of the tensor product  $ad \otimes ad$  into irreducible representations, and let  $P_i$  be the projector onto  $R_i$ . Then we have

$$\sum_a y^a y^a = \sum_i C_2(R_i) P_i, \quad \sum_a (t^a t^a \otimes \mathbf{1}) = \sum_a (\mathbf{1} \otimes t^a t^a) = C_2(ad) (\mathbf{1} \otimes \mathbf{1}) \tag{5.29}$$

and hence  $Q = \sum_i \lambda_i P_i$ , where

$$\lambda_i = \frac{C_2(R_i)}{2} - C_2(ad) \tag{5.30}$$

are the eigenvalues of  $Q$ , so that

$$C_l^{(\mathfrak{g})}(ad(\mathfrak{g})) = \frac{\text{Tr}_{ad \otimes ad}(Q^l)}{\dim(\mathfrak{g})} = \sum_i \frac{\dim(R_i)}{\dim(\mathfrak{g})} \lambda_i^l. \tag{5.31}$$

The eigenvalues of  $Q$  for all the simple Lie algebras are listed in table 6. For example for the  $e8$  and  $d16$  algebras, the operator  $Q$  equals

$$\begin{aligned} e8 : \quad Q &= -60 \cdot P_1 - 30 \cdot P_{248} - 12 \cdot P_{3875} + 0 \cdot P_{30380} + 2 \cdot P_{27000}, \\ d16 : \quad Q &= -60 \cdot P_1 - 30 \cdot P_{496} - 28 \cdot P_{527} - 4 \cdot P_{35960} + 0 \cdot P_{122264} + 2 \cdot P_{86768}, \end{aligned}$$

where we have labelled the different projectors  $P_i$  by the dimension of  $R_i$ . Together with the equations (5.26) and (5.31) this then leads to (5.14).

Now we can prove our claim (5.12). If  $\mathfrak{g} = u(1)$ , then all  $\xi_l(\mathfrak{g}, k) = 0$ , and thus also  $\mathfrak{g}' = u(1)$ . Otherwise, if  $\xi_l(\mathfrak{g}, k) = \xi_l(\mathfrak{g}', k')$  for all  $l$ , then this implies, because of (5.31), that all the eigenvalues of  $Q/k$  and  $Q'/k'$  must agree — the factors  $\dim(\mathfrak{g})$  and  $\dim(\mathfrak{g}')$  only affect the multiplicities of such eigenvalues. But this then implies that all the eigenvalues  $\lambda_i$  of  $Q$  and  $\lambda'_i$  of  $Q'$  must be related as

$$\lambda_i = \frac{k}{k'} \lambda'_i . \tag{5.32}$$

Each simple Lie algebra has a unique positive eigenvalue equal to 2, and thus (5.32) can only be satisfied if  $k = k'$ . But then (5.32) requires that the eigenvalues of  $Q$  and  $Q'$  are the same, but it is immediate from table 6 that this is only possible if  $\mathfrak{g} = \mathfrak{g}'$ . Thus  $\xi_l(\mathfrak{g}, k) = \xi_l(\mathfrak{g}', k')$  for all  $l$  implies that  $\mathfrak{g} = \mathfrak{g}'$  and  $k = k'$ . This completes our proof.

### 5.3 Identifying representations

In the previous section we have seen that we can determine the affine algebra symmetry of a meromorphic conformal field theory from its vacuum amplitudes. An obvious refinement of this question is whether we can similarly determine the representation content of the theory.

To answer this question we proceed in the same manner as before, except that we now take  $h$  in (5.8) to assume any value, not just  $h = 1$ . Since  $\dim(\mathcal{H}_h)$  is finite, only a finite set of irreducible representations of the Lie algebra  $\mathfrak{g}$  can appear in the decomposition  $\mathcal{H}_h = \oplus_i R_i$ . Furthermore, by the same arguments as in section 5.2, the vacuum amplitudes determine the trace over  $\mathcal{H}_h$  of any polynomial in the Casimir operators  $C_l^{(\mathfrak{g})}$ . Using the same techniques as above, the question of whether we can determine the representation content uniquely then boils down to the question of whether we can distinguish all representations  $R_i$  by their eigenvalues with respect to the Casimir operators  $C_l^{(\mathfrak{g})}$ . In the following we shall assume that  $\mathfrak{g}$  is simple; we shall come back to question of how to deal with the semi-simple case in section 5.4.

It is well known that we can distinguish the representations of any simple Lie algebra  $\mathfrak{g}$  by the eigenvalues of *all invariants*. The algebra of invariants of a simple Lie algebra  $\mathfrak{g}$  is generated by a set of  $\text{rank}(\mathfrak{g})$  Casimir operators

$$C_l^\perp := c^{a_1 \dots a_l} t^{a_1} \dots t^{a_l} , \tag{5.33}$$

where  $l$  takes values in a finite set of degrees that depends on the Lie algebra  $\mathfrak{g}$  in question, and we are using again the orthonormal basis with respect to the Killing form — see (2.8). The tensors  $c^{a_1 \dots a_l}$  can be taken to be totally symmetric in the indices, and to satisfy an orthonormality condition

$$c^{a_1 \dots a_l} c^{a_1 \dots a_l} = 1, \quad c^{a_1 \dots a_l} c^{a_1 \dots a_{l'}} = 0, \quad \text{if } l' \neq l. \tag{5.34}$$

The Casimir operators  $C_l^{(\mathfrak{g})}$  we have used above (see (5.5)) can obviously be expressed in terms of these generators as

$$C_l^{(\mathfrak{g})} = I_l(\mathfrak{g}) C_l^\perp + \text{polynomial in Casimirs of lower degree} . \tag{5.35}$$

Using the orthonormality condition (5.34), the index  $I_l(\mathfrak{g})$  turns out to be

$$I_l(\mathfrak{g}) = \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) c^{a_1 \dots a_l} = \dim(\mathfrak{g}) C_l^\perp(ad(\mathfrak{g})) . \quad (5.36)$$

This allows us to determine the subalgebra generated by the  $C_l^{(\mathfrak{g})}$  in principle.

It is not difficult to see that the Casimirs  $C_l^{(\mathfrak{g})}$  agree on two representations that are related to one another by an (outer) automorphism of the Lie algebra. Thus it is clear that we cannot distinguish between two representations that are related to one another in this way. However, it is natural to conjecture (and we have circumstantial evidence for it — see appendix C), that this is the only ambiguity:

**Conjecture.** If  $R_1$  and  $R_2$  are two irreducible representations of a simple Lie algebra  $\mathfrak{g}$  such that the eigenvalues of  $C_l^{(\mathfrak{g})}$  on  $R_i$  are equal,

$$C_l^{(\mathfrak{g})}(R_1) = C_l^{(\mathfrak{g})}(R_2) \quad \text{for all } l \quad (5.37)$$

then either  $R_1 \cong R_2$  or  $R_1 \cong \pi(R_2)$ , where  $\pi$  is a non-trivial (outer) automorphism of  $\mathfrak{g}$ .

For the simple Lie algebras (that we are currently considering) the only non-trivial outer automorphisms are charge conjugation for  $a(r)$ ,  $e6$  and  $d(r)$  with  $r$  odd. For  $d(r)$  with  $r$  even, the outer automorphism changes the chirality (spin flip) but does not map a representation to its conjugate. Finally, there is the special case of  $d4 = so(8)$ , for which there is ‘triality’.

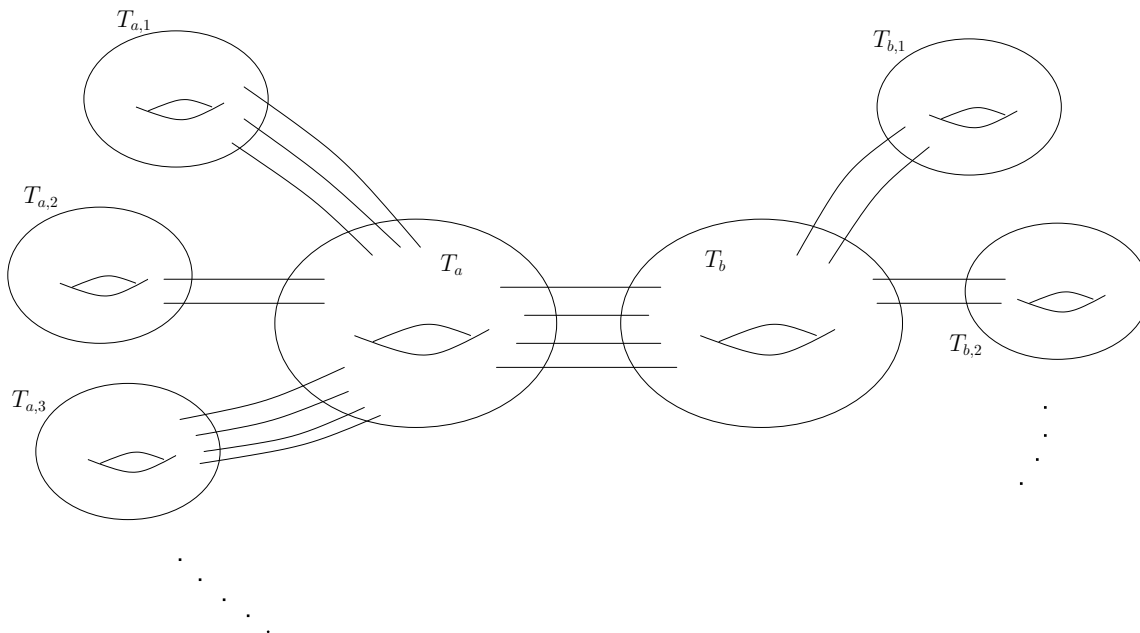
If the conjecture is true, then our analysis allows us (for  $\mathfrak{g}$  simple) to identify the representation content at each conformal weight up these automorphisms. Since the actual spectrum has to be real, we know on the other hand, that all representations must appear in complex conjugate pairs. Thus the ambiguity related to charge conjugation is irrelevant. The only genuine ambiguity then occurs for the case of  $d(n)$  with  $n$  even, where our analysis does not let us distinguish between representations of the opposite chirality; for  $d(4)$  there is in addition triality.

Obviously an *overall spin-flip* relates isomorphic conformal field theories to one another, and we therefore should not be able to distinguish such theories. However, on the basis of our present analysis we have not yet shown that the ambiguity is just an overall spin-flip. In particular, we cannot yet distinguish between two conformal field theories for which, say  $\mathcal{H}_h^{(1)} = S_+ \oplus S_-$  and  $\mathcal{H}_h^{(2)} = S_+ \oplus S_+$  for some  $h$ , where  $S_\pm$  describe spinor representations of opposite chirality. We shall come back to this point in section 5.4.2.

#### 5.4 Other degeneration limits

There are two issues that remain to be discussed: first the question of how to deal with semi-simple Lie algebras (see the discussion at the beginning of section 5.3); and secondly the question of how to show that the spin flip ambiguity is only an overall ambiguity (see the end of previous section). Both of these questions can be addressed by considering more general degeneration limits of the type depicted in figure 5. We shall not attempt to develop the general theory, but our arguments below will suggest how both problems can be solved using such techniques.





**Figure 5.** A more general degeneration limit. We are interested in the expansion where the modular parameters of the tori  $T_{a,1}, T_{a,2}, \dots, T_{b,1}, \dots$  are taken to linear order, while we consider the power  $q_a^{h_a} q_b^{h_b}$  for the modular parameters of the two tori  $T_a$  and  $T_b$ .

**5.4.1 Direct sums of algebras**

Up to now we have implicitly discussed the case where  $\mathfrak{g}$  is a simple affine algebra. The situation where  $\hat{\mathfrak{g}} = \oplus_i n_i \hat{\mathfrak{g}}_i$  can be dealt with similarly. Recall from section 5 that we can define polynomial Lie algebra invariants  $P_i$  that act in  $\mathcal{H}_1$  as a projector onto the subalgebra  $\mathfrak{g}_i$ . By taking  $h_a = 1$  with  $h_b$  arbitrary, as well as the modular parameters of the nodes between the tori  $T_a$  and  $T_b$  to be at linear order, we can obtain from the above degeneration limit (see figure 5) the invariant

$$\text{Tr}_{\mathcal{H}_1}(P_i \hat{t}^{a_1} \dots \hat{t}^{a_l}) \text{Tr}_{\mathcal{H}_h}(\hat{t}^{a_1} \dots \hat{t}^{a_l}) . \tag{5.38}$$

The first trace is only non-zero, if all  $t^{a_j}$  lie in  $\mathfrak{g}_i$ , and thus we can identify the representation content with respect to this Lie algebra separately from the rest. Using the techniques from the previous section, this allows us to deal with the case where all  $n_i = 1$ .

If some affine Lie algebra appears with higher multiplicity, the situation is more complicated. However, this has to be so since theories with a non-trivial multiplicity also have a bigger outer automorphism symmetry, namely the permutation symmetry that exchanges the different copies of  $\hat{\mathfrak{g}}_i$ .

**5.4.2 Spin flipped representations for  $d(r)$  with  $r$  even**

As we explained above, so far we cannot distinguish between theories  $\mathcal{H}_h = m_+ S_+ \oplus m_- S_-$  with different values for  $(m_+, m_-)$ . In fact, the techniques of the previous section only allow us to determine  $m_+ + m_-$ . We now want to show how we can also determine  $(m_+ - m_-)^2$ .

(We should not be able to determine directly  $(m_+ - m_-)$  since the overall spin-flip exchanges  $m_+$  and  $m_-$  and hence changes the sign of  $(m_+ - m_-)$ .)

To this end we now consider the degeneration limit of figure 5 with  $h_a = h_b = h$ . Furthermore we consider combinations of such configurations for which the external tori  $(T_{a_1}, T_{a_2}, \dots, T_{b_1}, \dots)$  generate projectors  $P_S$  onto  $m_+ S_+ \oplus m_- S_-$  — this is possible since the Casimirs  $C_l^{(\mathfrak{g})}$  allow us to define such projectors. Thus we can obtain the invariant

$$\text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) \text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) . \tag{5.39}$$

This product of traces can be decomposed as

$$\text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) \text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) = a \text{Tr}_{\mathcal{H}_h}(P_S \tilde{C}_r^\perp) \text{Tr}_{\mathcal{H}_h}(P_S \tilde{C}_r^\perp) + \dots , \tag{5.40}$$

where  $a$  is a non-zero coefficient which can be explicitly computed and the ellipses denote the terms corresponding to polynomials of degree  $r$  in  $C_l^\perp$  with  $l < r$ . These terms can be computed explicitly and depend on the eigenvalues  $C_l^{\mathfrak{g}}(S^\pm)$  and on the sum  $m_+ + m_-$  of the multiplicities. Thus we obtain

$$\text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) \text{Tr}_{\mathcal{H}_h}(P_S t^{a_1} \dots t^{a_r}) = a \dim(S^\pm)^2 |\tilde{C}_r^\perp(S^\pm)|^2 (m_+ - m_-)^2 + \dots . \tag{5.41}$$

Thus we can indeed determine  $(m_+ - m_-)^2$ .

It should similarly be possible to determine the relative chiralities at different conformal weights, simply by repeating the argument for  $h_a \neq h_b$ . In this way one should be able to show that the vacuum amplitudes allow one to identify these theories up to an overall spin flip.

## 6 Conclusions

In this paper we have studied the question of whether a conformal field theory is uniquely characterised by its higher genus vacuum amplitudes. For the case of a meromorphic (chiral) conformal field theory we have shown that the affine Lie algebra symmetry (that is generated by the currents at  $h = 1$ ) can be determined uniquely from the higher genus vacuum amplitudes. We have also given strong arguments that suggest that the vacuum amplitudes specify the representation content of the theory (with respect to this affine algebra), up to an overall automorphism of the finite Lie algebra.

We have applied our general arguments to some simple interesting examples, in particular the self-dual theories at  $c = 16$  and  $c = 24$ . Among other things this has allowed us to give an elementary proof that the  $E_8 \times E_8$  and the  $\text{Spin}(32)/\mathbb{Z}_2$  theories at  $c = 16$  have different genus  $g = 5$  vacuum amplitudes. The fact that the discrepancy only occurs at a rather high genus is a consequence of the modular properties of higher genus amplitudes at small values of the central charge. In particular, at  $c \leq 24$  the genus one amplitude already determines the amplitudes for genus  $g \leq 4$  uniquely. On the other hand, at  $c = 32$ , the different theories have typically already different genus  $g = 2$  amplitudes.

For ease of notation we considered only meromorphic (chiral) theories in this paper. It should be fairly obvious how to reformulate our arguments in the general case. In

particular, the analogue of (2.3) will in general be a power series in  $q_j^{h_j} \bar{q}_j^{\bar{h}_j}$ , and we can thus pick out the contribution from the states with arbitrary left- and right-moving conformal weights  $(h_j, \bar{h}_j)$ . For example, in order to determine the left-moving affine symmetry, we can consider the terms that go as  $q_j^1 \bar{q}_j^0$ , etc., and the analysis is then essentially the same as in the meromorphic context. Similarly, the representation content can be determined with respect to both left- and right-moving affine algebras, up to separate automorphisms of the left- and right-moving Lie algebra.

Our arguments thus go a certain way towards showing that a conformal field theory is uniquely determined by its vacuum amplitudes. However, it should be clear that they do not settle the question completely. In particular, we cannot say much about theories without any current symmetries, such as for example the Monster theory, although similar techniques will clearly also constrain these theories. It would be interesting to gain insight into this question, in particular in connection with the conjectured uniqueness of the Monster theory.

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### A Decomposition

For the calculation of the trace over  $\mathcal{H}_2$  of the powers of the quadratic Casimir  $C_2^l$  in section 3.2 it is important to know the decomposition of  $\mathcal{H}_2$  with respect to  $\mathfrak{g}$ . It is useful to decompose  $\mathcal{H}_2$  as

$$\mathcal{H}_2 = \mathcal{H}_2^{(0)} \oplus \mathcal{H}_2^{\text{hw}}, \tag{A.1}$$

where  $\mathcal{H}_2^{(0)}$  are the states at conformal weight two in the vacuum representation of the affine Lie algebra  $\hat{\mathfrak{g}}$ , while  $\mathcal{H}_2^{\text{hw}}$  are the states that are highest weight with respect to the affine Lie algebra. The states in the vacuum representation can be determined using the decomposition of the tensor products of the adjoint. This leads to

$$\begin{aligned} d16 e8 : \quad \mathcal{H}_2^{(0)} &= 2 \cdot (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{496}, \mathbf{1})_{60} \oplus (\mathbf{1}, \mathbf{248})_{60} \oplus (\mathbf{496}, \mathbf{248})_{120} \oplus (\mathbf{527}, \mathbf{1})_{64} \\ &\quad \oplus (\mathbf{35960}, \mathbf{1})_{112} \oplus (\mathbf{1}, \mathbf{3875})_{96} \\ (e8)^3 : \quad \mathcal{H}_2^{(0)} &= 3 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{248}, \mathbf{1}, \mathbf{1})_{60} \oplus (\mathbf{3875}, \mathbf{1}, \mathbf{1})_{96} \oplus (\mathbf{248}, \mathbf{248}, \mathbf{1})_{120} \\ &\quad \oplus (\text{cycl. perm.}) \end{aligned}$$

$$\begin{aligned}
 a17 e7 : \quad \mathcal{H}_2^{(0)} &= 2 \cdot (\mathbf{1}, \mathbf{1})_0 \oplus 2 \cdot (\mathbf{323}, \mathbf{1})_{36} \oplus (\mathbf{1}, \mathbf{133})_{36} \oplus (\mathbf{323}, \mathbf{133})_{72} \\
 &\quad \oplus (\mathbf{23085}, \mathbf{1})_{68} \oplus (\mathbf{1}, \mathbf{1539})_{56} \\
 d10 (e7)^2 : \quad \mathcal{H}_2^{(0)} &= 3 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{190}, \mathbf{1}, \mathbf{1})_{36} \oplus (\mathbf{1}, \mathbf{133}, \mathbf{1})_{36} \oplus (\mathbf{190}, \mathbf{133}, \mathbf{1})_{72} \\
 &\quad \oplus (\mathbf{1}, \mathbf{133}, \mathbf{133})_{72} \oplus (\mathbf{209}, \mathbf{1}, \mathbf{1})_{40} \oplus (\mathbf{4845}, \mathbf{1}, \mathbf{1})_{64} \oplus (\mathbf{1}, \mathbf{1539}, \mathbf{1})_{56} \\
 &\quad \oplus (2 \leftrightarrow 3) \\
 a11 d7 e6 : \quad \mathcal{H}_2^{(0)} &= 3 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus 2 \cdot (\mathbf{143}, \mathbf{1}, \mathbf{1})_{24} \oplus (\mathbf{1}, \mathbf{91}, \mathbf{1})_{24} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{78})_{24} \\
 &\quad \oplus (\mathbf{143}, \mathbf{91}, \mathbf{1})_{48} \oplus (\mathbf{143}, \mathbf{1}, \mathbf{78})_{48} \oplus (\mathbf{1}, \mathbf{91}, \mathbf{78})_{48} \oplus (\mathbf{4212}, \mathbf{1}, \mathbf{1})_{44} \\
 &\quad \oplus (\mathbf{1}, \mathbf{104}, \mathbf{1})_{28} \oplus (\mathbf{1}, \mathbf{1001}, \mathbf{1})_{40} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{650})_{36} \\
 (e6)^4 : \quad \mathcal{H}_2^{(0)} &= 4 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{78}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{24} \oplus (\mathbf{78}, \mathbf{78}, \mathbf{1}, \mathbf{1})_{48} \oplus (\mathbf{650}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{36} \\
 &\quad \oplus (\text{perm.}) \\
 (a9)^2 d6 : \quad \mathcal{H}_2^{(0)} &= 3 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus 2 \cdot (\mathbf{99}, \mathbf{1}, \mathbf{1})_{20} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{66})_{20} \oplus (\mathbf{99}, \mathbf{99}, \mathbf{1})_{40} \\
 &\quad \oplus (\mathbf{99}, \mathbf{1}, \mathbf{66})_{40} \oplus (\mathbf{1925}, \mathbf{1}, \mathbf{1})_{36} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{77})_{24} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{495})_{32} \\
 &\quad \oplus (1 \leftrightarrow 2) \\
 (d6)^4 : \quad \mathcal{H}_2^{(0)} &= 4 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{66}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{20} \oplus (\mathbf{66}, \mathbf{66}, \mathbf{1}, \mathbf{1})_{40} \oplus (\mathbf{77}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{24} \\
 &\quad \oplus (\mathbf{495}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{32} \oplus (\text{perm.}) \\
 (a5)^4 d4 : \quad \mathcal{H}_2^{(0)} &= 5 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus 2 \cdot (\mathbf{35}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{12} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{28})_{12} \\
 &\quad \oplus (\mathbf{35}, \mathbf{35}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{24} \oplus (\mathbf{35}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{28})_{24} \oplus (\mathbf{189}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{20} \\
 &\quad \oplus 3 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{35})_{16} \oplus (\text{perm. } \{1, 2, 3, 4\}) \\
 (d4)^6 : \quad \mathcal{H}_2^{(0)} &= 6 \cdot (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{28}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{12} \oplus (\mathbf{28}, \mathbf{28}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{24} \\
 &\quad \oplus 3 \cdot (\mathbf{35}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{16} \oplus (\text{perm.}), \tag{A.2}
 \end{aligned}$$

where the index always denotes the value of the quadratic Casimir.

To determine the contribution from  $\mathcal{H}_2^{\text{hw}}$  we recall that for each simple Lie algebra  $\mathfrak{g}$ , the Sugawara construction gives

$$L_0 = \frac{1}{2(k + h^\vee(\mathfrak{g}))} \left( C_2 + 2 \sum_{n=1}^{\infty} J_{-n}^a J_n^a \right), \tag{A.3}$$

and thus on highest weight states we have

$$C_2 = 2(k + h^\vee(\mathfrak{g}))L_0. \tag{A.4}$$

Since the highest weight states in  $\mathcal{H}_2^{\text{hw}}$  have conformal dimension  $h = 2$ , it thus follows that

$$C_2(\mathcal{H}_2^{\text{hw}}) = 4(k + h^\vee(\mathfrak{g})). \tag{A.5}$$

For a semi-simple Lie algebra  $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ , this reasoning has to be applied to each factor separately, but the situation is particularly simple if all  $k_i$  and all  $h^\vee(\mathfrak{g}_i)$  are the same, as

is the case for the lattice theories at  $c = 24$ . For these theories also the dimension of  $\mathcal{H}_2^{\text{hw}}$  can be easily determined, since one knows that the total dimension  $\dim \mathcal{H}_2 = 196884$ , and the dimension of  $\mathcal{H}_2^{(0)}$  can be determined as above. This leads to

$$\begin{aligned}
 d16\ e8 : \dim \mathcal{H}_2^{\text{hw}} &= 32768 & C_2(\mathcal{H}_2^{\text{hw}}) &= 124 \\
 (e8)^3 : \dim \mathcal{H}_2^{\text{hw}} &= 0 \\
 a17\ e7 : \dim \mathcal{H}_2^{\text{hw}} &= 128520 & C_2(\mathcal{H}_2^{\text{hw}}) &= 76 \\
 d10\ (e7)^2 : \dim \mathcal{H}_2^{\text{hw}} &= 120064 & C_2(\mathcal{H}_2^{\text{hw}}) &= 76 \\
 a11\ d7\ e6 : \dim \mathcal{H}_2^{\text{hw}} &= 159194 & C_2(\mathcal{H}_2^{\text{hw}}) &= 52 \\
 (e6)^4 : \dim \mathcal{H}_2^{\text{hw}} &= 157464 & C_2(\mathcal{H}_2^{\text{hw}}) &= 52 \\
 (a9)^2\ d6 : \dim \mathcal{H}_2^{\text{hw}} &= 169128 & C_2(\mathcal{H}_2^{\text{hw}}) &= 44 \\
 (d6)^4 : \dim \mathcal{H}_2^{\text{hw}} &= 168192 & C_2(\mathcal{H}_2^{\text{hw}}) &= 44 \\
 (a5)^4\ d4 : \dim \mathcal{H}_2^{\text{hw}} &= 184440 & C_2(\mathcal{H}_2^{\text{hw}}) &= 28 \\
 (d4)^6 : \dim \mathcal{H}_2^{\text{hw}} &= 184320 & C_2(\mathcal{H}_2^{\text{hw}}) &= 28 .
 \end{aligned} \tag{A.6}$$

With this information it is then straightforward to determine the trace of the powers of the quadratic Casimir; for example, we have

$$\begin{aligned}
 d16\ e8 : \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) &= \left[ 2 \cdot 0^l + (496 + 248) \cdot 60^l + (496 \cdot 248) \cdot 120^l \right. \\
 &\quad \left. + (527 \cdot 64^l + 35960 \cdot 112^l) + (3875 \cdot 96^l) \right] + \left[ 32768 \cdot 124^l \right] \\
 (e8)^3 : \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) &= \left[ 3 \cdot 0^l + (3 \cdot 248) \cdot 60^l + 3 \cdot (248 \cdot 248) \cdot 120^l + 3 \cdot (3875 \cdot 96^l) \right] \\
 a17\ e7 : \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) &= \left[ 2 \cdot 0^l + (323 + 133) \cdot 36^l + (323 \cdot 133) \cdot 72^l \right. \\
 &\quad \left. + (323 \cdot 36^l + 23085 \cdot 68^l) + (1539 \cdot 56^l) \right] + \left[ 128520 \cdot 76^l \right] \\
 d10\ (e7)^2 : \text{Tr}_{\mathcal{H}_2}(C_2^l) &= \left[ 3 \cdot 0^l + (190 + 2 \cdot 133) \cdot 36^l + (2 \cdot 190 \cdot 133 + 133^2) \cdot 72^l \right. \\
 &\quad \left. + (209 \cdot 40^l + 4845 \cdot 64^l) + 2 \cdot (1539 \cdot 56^l) \right] + \left[ 120064 \cdot 76^l \right] .
 \end{aligned} \tag{A.7}$$

This then reproduces the results of table 2.

Finally, for the Leech lattice theory, the quadratic Casimir is just the length squared of the underlying lattice vector. At conformal dimension  $h = 2$ , of the 196884 states, 324 are descendants of the vacuum, while the remaining 196560 come from the lattice vectors of length squared 4. Thus for the Leech theory we simply have

$$\text{Leech:} \quad \text{Tr}_{\mathcal{H}_2}(C_2^l) = 324 \cdot 0^l + 196560 \cdot 4^l . \tag{A.8}$$

## B Riemann surfaces

### B.1 Riemann period matrices and modular forms

In order to analyse the modular properties of partition functions, it is useful to define the period matrix of a Riemann surface. Let  $\Sigma$  be a compact Riemann surface of genus  $g > 0$ .

Let us define a basis of the first homology group  $H_1(\Sigma, \mathbb{Z}) = \{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ , with symplectic intersection matrix

$$\#(\alpha_i, \alpha_j) = 0 = \#(\beta_i, \beta_j), \quad \#(\alpha_i, \beta_j) = \delta_{ij}, \quad i, j = 1, \dots, g. \quad (\text{B.1})$$

This condition determines the basis up to a symplectic transformation

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} := \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}), \quad (\text{B.2})$$

where  $\alpha$  and  $\beta$  are  $g$ -dimensional vectors, and  $A, B, C, D$  are  $g \times g$  matrices. The choice of such a basis uniquely determines a basis  $\{\omega_1, \dots, \omega_g\}$  of holomorphic 1-differentials normalised with respect to the  $\alpha$ -cycles

$$\oint_{\alpha_i} \omega_j = \delta_{ij}, \quad i, j = 1, \dots, g. \quad (\text{B.3})$$

The Riemann period matrix of  $\Sigma$  is then defined by

$$\Omega_{ij} = \oint_{\beta_i} \omega_j, \quad (\text{B.4})$$

and it has the properties

$$\Omega_{ij} = \Omega_{ji}, \quad \text{Im } \Omega > 0. \quad (\text{B.5})$$

Obviously, the basis  $\{\omega_1, \dots, \omega_g\}$ , and the Riemann period matrix depend on the choice of the symplectic basis of  $H_1(\Sigma, \mathbb{Z})$ ; under the action (B.2) of the symplectic group, the holomorphic 1-differentials transform as

$$(\omega_1, \dots, \omega_g) \mapsto (\tilde{\omega}_1, \dots, \tilde{\omega}_g) = (\omega_1, \dots, \omega_g)(C\Omega + D)^{-1}, \quad (\text{B.6a})$$

$$\Omega \mapsto \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}. \quad (\text{B.6b})$$

Let us define the Siegel upper half-space as the space of  $g \times g$  symmetric complex matrices with positive definite imaginary part,

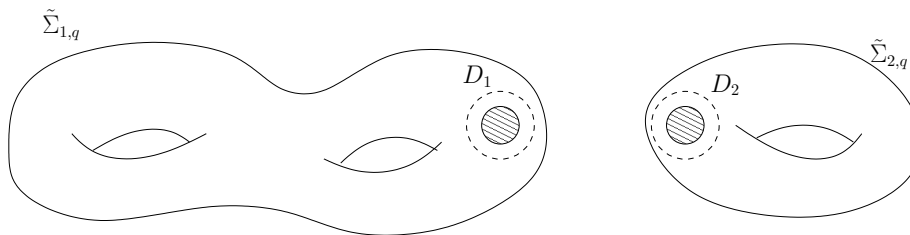
$$\mathfrak{H}_g = \{Z \in M_g(\mathbb{C}) \mid Z_{ij} = Z_{ji}, \text{Im } Z > 0\}. \quad (\text{B.7})$$

The locus  $\mathcal{J}_g \subseteq \mathfrak{H}_g$  of all the period matrices of genus  $g$  Riemann surfaces is dense in  $\mathfrak{H}_g$  for  $g \leq 3$ , whereas for  $g > 3$  its closure  $\bar{\mathcal{J}}_g$  is a  $(3g - 3)$ -dimensional subspace of  $\mathfrak{H}_g$ . The quotient  $\mathcal{J}_g / \text{Sp}(2g, \mathbb{Z})$  is isomorphic to  $\mathcal{M}_g$ ; in particular, the Riemann period matrices of two different Riemann surfaces lie in different  $\text{Sp}(2g, \mathbb{Z})$ -orbits in  $\mathcal{J}_g$ .

A (Siegel) modular form  $f$  of degree  $g$  and weight  $k$  is a holomorphic function on  $\mathfrak{H}_g$  such that

$$f((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k f(Z), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}). \quad (\text{B.8})$$

For  $g = 1$ , we also require that  $f$  is holomorphic at the cusps; a cusp is a fix-point  $p \in \mathbb{R} \cup \{\infty\}$  under the action of some  $M \in \text{Sp}(2, \mathbb{Z}) \cong \text{SL}(2, \mathbb{Z})$  with  $\text{Tr}(M) = \pm 2$  (a parabolic element). An analogous condition is automatically satisfied for  $g > 1$ .



**Figure 6.** The surfaces with boundary  $\Sigma_{1,q}$  and  $\Sigma_{2,q}$  are the complements of the discs of radius  $|q|^{1/2}$  (the filled discs in the picture) on  $\Sigma_1$  and  $\Sigma_2$ . The surface  $\Sigma_q$  is obtained by sewing together  $\Sigma_{1,q}$  and  $\Sigma_{2,q}$ , via the identification (B.10) along the boundaries of the discs. The dashed circles are the boundaries of the coordinate patches  $D_1$  and  $D_2$ .

### B.2 Degeneration limits and singular Riemann surfaces

The moduli space  $\mathcal{M}_g$  of smooth Riemann surfaces of genus  $g > 1$  is the quotient of the Teichmüller space, a complex topologically trivial space of dimension  $(3g - 3)$ , by the discrete mapping class group. The moduli space  $\mathcal{M}_g$  is not compact, and its Deligne-Mumford compactification  $\bar{\mathcal{M}}_g$  is obtained by adjoining Riemann surfaces whose only singularities are nodes. In fact, the boundary  $\partial\bar{\mathcal{M}}_g$  is the union of  $\lfloor g/2 \rfloor + 1$  divisors

$$\partial\bar{\mathcal{M}}_g = \Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_{\lfloor g/2 \rfloor}, \tag{B.9}$$

where a generic point of  $\Delta_k$  corresponds to a Riemann surface with a node linking two smooth connected components of genus  $k$  and  $g - k$ , respectively. ( $\Delta_0$  is the component where the node links two points on a single surface of genus  $g - 1$ ). In either case the singular surface is the limit  $\lim_{q \rightarrow 0} \Sigma_q$  in  $\bar{\mathcal{M}}_g$ , of a suitable family  $\{\Sigma_q\}_{0 < |q| < 1}$  of smooth Riemann surfaces, parametrised by a complex degeneration parameter  $q \in \mathbb{C}$ . The degenerating surface  $\Sigma_q$ ,  $|q| > 0$ , is defined by the standard plumbing fixture procedure (see for example [34]), where one identifies (for  $k > 0$ ) the boundaries of local discs via

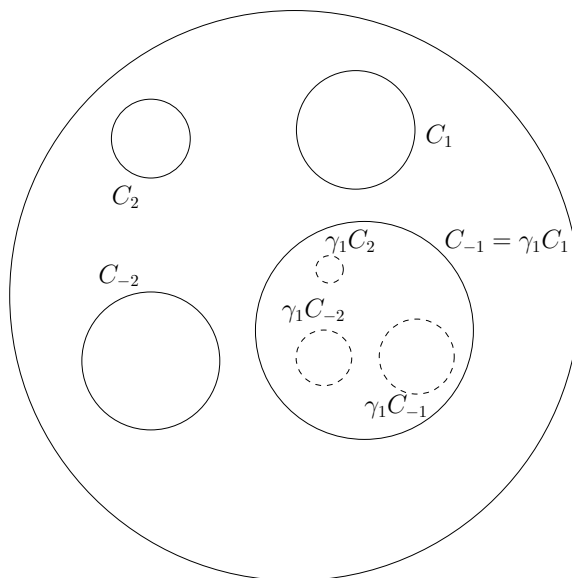
$$z_1(p_1) = \frac{q}{z_2(p_2)}. \tag{B.10}$$

Here  $z_i : D_i \rightarrow \mathbb{C}$  are the local coordinates on some  $D_i \subset \Sigma_i$ ,  $i = 1, 2$ , and (B.10) identifies the points  $p_i \in D_i$  on the circles  $|z_i(p_i)| = |q|^{1/2}$ ,  $i = 1, 2$  (see figure 6). In the limit  $q \rightarrow 0$ , the Riemann surface  $\Sigma_q$  degenerates to the singular surface obtained by joining  $\Sigma_1$  and  $\Sigma_2$ , with the points  $u \in \Sigma_1$  and  $v \in \Sigma_2$  (that lie at the centres of the discs  $D_1$  and  $D_2$ , respectively) identified to form a node.

For the case of  $\Delta_0$  the only difference is that  $u$  and  $v$  lie on the same Riemann surface of genus  $g - 1$ . Similarly, it is clear that we can also consider a family of smooth curves  $\{\Sigma_{q_1, \dots, q_n}\}$  depending on  $n$  degeneration parameters  $q_i$ ,  $0 < |q_i| < 1$ . As long as the points  $u_1, v_1, \dots, u_n, v_n$  are pairwise distinct, the limit  $\lim_{q_1, \dots, q_n \rightarrow 0} \Sigma_{q_1, \dots, q_n}$  is well defined and corresponds to a singular Riemann surface with  $n$  nodes.

### B.3 Schottky uniformisation

A convenient description of genus  $g$  Riemann surfaces can be given in terms of the Schottky uniformisation. Let  $D$  be the open subset of the Riemann sphere  $\hat{\mathbb{C}}$ , obtained by removing



**Figure 7.** Schottky uniformization of a Riemann surface of genus 2. The fundamental domain  $D \subset \hat{\mathbb{C}}$  is the complement of the disks bounded by  $C_1, C_{-1}, C_2, C_{-2}$ . The Riemann surface is obtained by sewing together  $C_1$  with  $C_{-1}$  and  $C_2$  with  $C_{-2}$ . The dashed circles are the images of the cycles  $C_{-1}, C_{-2}, C_2$  under the action of the generator  $\gamma_1$ , that maps  $C_1$  to  $C_{-1}$ . The outer circle represents the Riemann sphere  $\hat{\mathbb{C}}$ .

$2g$  closed disks, with circle boundaries  $C_{\pm 1}, \dots, C_{\pm g}$ , from  $\hat{\mathbb{C}}$  (see figure 7). In order to obtain from this a genus  $g$  surface, we want to identify the boundary component  $C_r$  with  $C_{-r}$ , for  $r = 1, \dots, g$ . More precisely, let us define  $g$  fractional linear transformations  $\gamma_1, \dots, \gamma_g \in \text{PSL}(2, \mathbb{Z})$ , such that  $\gamma_r$  maps  $C_r$  to  $C_{-r}$ , for each  $r = 1, \dots, g$ . We call the discrete subgroup  $\Gamma$  of  $\text{PSL}(2, \mathbb{Z})$  with distinguished free generators  $\gamma_1, \dots, \gamma_g$  the marked Schottky group. It is not difficult to see that  $D \subset \hat{\mathbb{C}}$  is a fundamental domain for  $\Gamma$ , and that  $\Sigma$  can be defined as the quotient of the Riemann sphere by  $\Gamma$ . (Strictly speaking, we have to exclude the limit points of fixed points of  $\Gamma$ .)

All elements of  $\Gamma$ , and in particular the generators  $\gamma_1, \dots, \gamma_g$ , are loxodromic, i.e. each  $\gamma \in \Gamma$  is conjugate in  $\text{PSL}(2, \mathbb{C})$  to the transformation  $z \mapsto qz$  for some multiplier  $q$ . The multiplier satisfies  $0 < |q| < 1$  and is uniquely determined by  $\gamma$ . More explicitly, we can therefore write  $\gamma_r(z)$  as

$$\frac{\gamma_r(z) - u_r}{\gamma_r(z) - v_r} = q_r \frac{z - u_r}{z - v_r}, \tag{B.11}$$

where  $0 < |q_r| < 1$ , and  $u_r, v_r \in \hat{\mathbb{C}}$  are the attracting and repelling fixed points of  $\gamma_r$ , respectively. Thus any marked Schottky group  $\Gamma$ , and subsequently any Riemann surface  $\Sigma = \Omega/\Gamma$ , is completely determined by specifying the multipliers and the attracting and repelling points of its generators. For  $g > 1$ , we can apply an overall  $\text{PSL}(2, \mathbb{C})$  conjugation to fix  $u_g = 0, v_g = \infty, u_{g-1} = 1$ ; the resulting Schottky group is called normalised. The space of normalised marked Schottky groups defines the Schottky space  $\mathfrak{S}$ . It is a  $(3g - 3)$ -



dimensional complex manifold parameterised by

$$q_1, \dots, q_g, u_1, \dots, u_{g-2}, v_1, \dots, v_{g-1}, \tag{B.12}$$

and it defines a finite covering of the moduli space of Riemann surfaces. The curves  $C_1, \dots, C_g$  can be taken to define the cycles  $\alpha_1, \dots, \alpha_g$  in a symplectic basis of  $H_1(\Sigma, \mathbb{Z})$  (see appendix B.1). It follows that the choice of a Schottky group uniformising a Riemann surface  $\Sigma$  canonically determines a basis  $\{\omega_1, \dots, \omega_g\}$  of holomorphic 1-differentials on  $\Sigma$ , satisfying the normalisation condition (B.3).

For  $g = 1$ , the Schottky group is a discrete abelian subgroup  $\Gamma \cong \mathbb{Z}$  of  $\text{PSL}(2, \mathbb{C})$ , freely generated by a loxodromic element  $\gamma$ . By a  $\text{PSL}(2, \mathbb{C})$ -conjugation the attracting and repelling points of  $\gamma$  can be fixed to 0 and  $\infty$  respectively, so that  $\gamma : z \mapsto qz$ , for some  $q \in \mathbb{C}$ ,  $0 < |q| < 1$ . The modular parameter  $\tau$  is related to  $q$  by  $q = e^{2\pi i\tau}$ ; the coordinate  $w$  on the usual torus  $w \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  is related to the coordinate  $z$  by

$$z(w) = q^{1/2} e^{2\pi i w}, \tag{B.13}$$

so that

$$z(w + 1) = z(w), \quad z(w + \tau) = \gamma(z(w)). \tag{B.14}$$

Finally, a family  $\Sigma_q$  of Riemann surfaces of genus  $g$  degenerating, in the limit  $q \rightarrow 0$ , to a singular surface in  $\Delta_0$ , can be easily described in terms of the Schottky uniformisation. Let us define a Schottky group  $\Gamma_q$  with generators  $\gamma_1, \dots, \gamma_{g-1}, \gamma_g(q)$  and such that the multiplier  $q_g$  of  $\gamma_g$  equals the degeneration parameter  $q_g = q$ . The limit  $q \rightarrow 0$  corresponds then to pinching the homologically non-trivial cycle  $C_g$  down to a point.

### C Evidence for the Lie algebra conjecture

Recall the conjecture of section 5.3: if  $R_1$  and  $R_2$  are two irreducible representations of a simple Lie algebra  $\mathfrak{g}$  such that the eigenvalues of  $C_l^{(\mathfrak{g})}$  on  $R_i$  are equal,

$$C_l^{(\mathfrak{g})}(R_1) = C_l^{(\mathfrak{g})}(R_2) \quad \text{for all } l, \tag{C.1}$$

where  $C_l^{(\mathfrak{g})}$  is the Casimir operator defined in (5.5), then either  $R_1 \cong R_2$  or  $R_1 \cong \pi(R_2)$ , where  $\pi$  is a non-trivial (outer) automorphism of  $\mathfrak{g}$ .

Let us collect some support for this conjecture. The situation is obviously simplest if the algebra generated by the  $C_l^{(\mathfrak{g})}$  is equivalent to the algebra generated by the  $C_l^\perp$ . Then the usual analysis for the invariant algebra shows that (C.1) implies  $R_1 \cong R_2$ .

The two algebras are the same if all  $I_l(\mathfrak{g}) \neq 0$  and if the different Casimir operators  $C_l^\perp$  have different degrees. Indeed then we can use (5.35) recursively to express the generators  $C_l^\perp$  in terms of  $C_l^{(\mathfrak{g})}$ , thus establishing that the algebra generated by the  $C_l^\perp$  is a subalgebra of the algebra generated by the  $C_l^{(\mathfrak{g})}$ , and hence isomorphic to it. The above condition is satisfied for the simple Lie algebras  $b(r)$ ,  $c(r)$ ,  $e7$ ,  $e8$ ,  $f4$  and  $g2$ . All of them do not have any non-trivial outer automorphisms, and thus  $R_1 \cong R_2$  is the only possibility. The other cases are more difficult, so let us deal with them in turn.

**C.1  $d(r)$  algebras**

For the  $d(r)$  algebras, the independent Casimirs have degrees  $2, 4, 6, \dots, 2r - 2, r$ . The analysis depends a bit on whether  $r$  is even or odd.

**$r$  odd.** If  $r$  is odd, then all the Casimir operators  $C_l^\perp$  have different degree, but for the Casimir of odd degree  $r$  the index  $I_r(\mathfrak{g})$  vanishes. In fact, the index always vanishes for Casimir operators of odd degree since the generators of the adjoint representations are anti-symmetric,  $t^a = -(t^a)^T$ , and thus

$$\begin{aligned} \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) &= (-1)^l \text{Tr}_{ad}((t^{a_1})^T \dots (t^{a_l})^T) = (-1)^l \text{Tr}_{ad}((t^{a_1} \dots t^{a_l})^T) \\ &= (-1)^l \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) . \end{aligned} \tag{C.2}$$

Since  $c^{a_1 \dots a_l}$  is totally symmetric, it then follows that

$$c^{a_1 \dots a_l} \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) = c^{a_1 \dots a_l} \text{Tr}_{ad}(t^{a_1} \dots t^{a_1}) = (-1)^l c^{a_1 \dots a_l} \text{Tr}_{ad}(t^{a_1} \dots t^{a_l}) , \tag{C.3}$$

thus showing that the index  $I_l(\mathfrak{g})$  vanishes if  $l$  is odd.

For the case of  $d(r)$  one can show by an explicit calculation that the algebra generated by the  $C_l^{(\mathfrak{g})}$  coincides with the subalgebra of the invariant algebra generated by

$$C_2^\perp, \dots, C_{2r-2}^\perp, (C_r^\perp)^2 . \tag{C.4}$$

This allows us to distinguish all representations, except those that differ by the sign of the eigenvalue of  $C_r^\perp$ . One can show that two representations that only differ by the sign of the eigenvalue of  $C_r^\perp$  are precisely charge conjugate representations. Thus we can identify representations up to charge conjugation, in agreement with the conjecture.

**$r$  even.** For  $r$  even, all the Casimir operators have even degree, but there are now two independent Casimirs of degree  $r$ , which we denote by  $C_r^\perp$  and  $\tilde{C}_r^\perp$ . We choose the convention that the invariant  $\tilde{C}_r^\perp$  of degree  $r$  is only non-zero for the spinor representations, i.e. the representations that are not representations of  $\text{SO}(2r)$ . It then follows that  $\tilde{I}_r(\mathfrak{g}) = 0$ , whereas it can be shown that the index  $I_l(\mathfrak{g})$ ,  $l = 2, \dots, 2r - 2$ , is related to the analogous index  $I_l(V)$  for the vector representation by

$$I_l(\mathfrak{g}) = (2r - 2^{l-1})I_l(V) . \tag{C.5}$$

It is known that  $I_l(V) \neq 0$  for all  $l = 2, \dots, 2r - 2$ , so that, if we restrict to the case where  $r$  is not an even power of 2, we obtain  $I_l(\mathfrak{g}) \neq 0$  as well. Provided that  $r \neq 4^n$  one can then show that the algebra generated by the  $C_l(\mathfrak{g})$  coincides with the subalgebra of the invariant algebra generated by

$$C_2^\perp, \dots, C_{2r-2}^\perp, (\tilde{C}_r^\perp)^2 . \tag{C.6}$$

This allows one to distinguish all representations, except those that differ by the sign of the eigenvalue of  $\tilde{C}_r^\perp$ , i.e. up to the outer automorphism corresponding to spin flip.

The case  $r = 4^n$  includes in particular  $d4 = so(8)$ , where we know that something special has to happen (since this algebra has an enhanced triality symmetry). In fact,

for  $d4$ , both the fourth order indices  $I_4(\mathfrak{g})$  and  $\tilde{I}_4(\mathfrak{g})$  vanish. Unfortunately, we have not been able to show that for  $r = 4^n$  with  $r \neq 4$ , the algebra generated by  $C_l^{\mathfrak{g}}$  is sufficient to distinguish irreducible representations up to spin flip. (However, we are also not aware of any counterexample.)

### C.2 $e6$ algebra

For the  $e6$  algebra, the degrees of the independent Casimirs are 2, 5, 6, 8, 9, 12. The indices  $I_l(\mathfrak{g})$  are non-zero for all the even  $l$ . One can show that the subalgebra generated by the  $C_l^{\mathfrak{g}}$  is precisely the subalgebra of the full invariant algebra generated by

$$C_2^{\perp}, C_6^{\perp}, C_8^{\perp}, (C_5^{\perp})^2, C_{12}^{\perp}, C_5^{\perp}C_9^{\perp}, (C_9^{\perp})^2. \tag{C.7}$$

This allows one to identify all representations up to charge conjugation.

### C.3 $a(r)$ algebras

The case of the  $a(r)$  algebras is the most complicated, because there are several Casimirs of odd degree. More precisely, the independent Casimirs have degree 2, 3, 4,  $\dots$ ,  $r + 1$ ; the index  $I_l(\mathfrak{g})$  of all the Casimirs of even degree is non-zero, but because of (C.3)  $I_l(\mathfrak{g}) = 0$  for all odd  $l$ . In analogy with the  $d(r)$  and  $e6$  cases, it is natural to expect that the subalgebra generated by the  $C_l^{(\mathfrak{g})}$  contains

$$C_2^{\perp}, C_4^{\perp}, C_6^{\perp}, \dots, C_{2\lfloor(r+1)/2\rfloor}^{\perp}, (C_3^{\perp})^2, C_3^{\perp}C_5^{\perp}, \dots, C_3^{\perp}C_{2\lfloor r/2\rfloor+1}^{\perp}. \tag{C.8}$$

This can be proved for  $r \leq 4$ , but we have not managed to establish it in general. If true, it would imply that we can identify representations up to charge conjugation.

## References

- [1] E. Witten, *Three-Dimensional Gravity Revisited*, [arXiv:0706.3359](#) [SPIRES].
- [2] A. Maloney and E. Witten, *Quantum Gravity Partition Functions in Three Dimensions*, [arXiv:0712.0155](#) [SPIRES].
- [3] D. Gaiotto and X. Yin, *Genus Two Partition Functions of Extremal Conformal Field Theories*, *JHEP* **08** (2007) 029 [[arXiv:0707.3437](#)] [SPIRES].
- [4] X. Yin, *Partition Functions of Three-Dimensional Pure Gravity*, [arXiv:0710.2129](#) [SPIRES].
- [5] X. Yin, *On Non-handlebody Instantons in 3D Gravity*, *JHEP* **09** (2008) 120 [[arXiv:0711.2803](#)] [SPIRES].
- [6] S. Grushevsky and R. Salvati Manni, *On the cosmological constant for the chiral superstring measure*, [arXiv:0809.1391](#) [SPIRES].
- [7] L. Dolan, P. Goddard and P. Montague, *Conformal field theory of twisted vertex operators*, *Nucl. Phys. B* **338** (1990) 529 [SPIRES]; *Conformal field theory, triality and the Monster group*, *Phys. Lett. B* **236** (1990) 165 [SPIRES].
- [8] A.N. Schellekens, *Meromorphic  $c = 24$  conformal field theories*, *Commun. Math. Phys.* **153** (1993) 159 [[hep-th/9205072](#)] [SPIRES].

- [9] D. Friedan and S.H. Shenker, *The Analytic Geometry of Two-Dimensional Conformal Field Theory*, *Nucl. Phys. B* **281** (1987) 509 [SPIRES].
- [10] D. Bernard, *On the Wess-Zumino-Witten models on Riemann surfaces*, *Nucl. Phys. B* **309** (1988) 145 [SPIRES].
- [11] G.B. Segal, *The definition of conformal field theory*, in *Topology, geometry and quantum field theory*, volume **308** of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, U.K. (2004), pg. 421–577 [SPIRES].
- [12] Y.C. Zhu, *Global vertex operators on Riemann surfaces*, *Commun. Math. Phys.* **165** (1994) 485 [SPIRES].
- [13] K. Gawędzki, *SU(2) WZW theory at higher genera*, *Commun. Math. Phys.* **169** (1995) 329 [[hep-th/9402091](#)] [SPIRES].
- [14] M.P. Tuite, *Genus two meromorphic conformal field theory*, presented at *Moonshine Workshop*, Montréal, Canada, May 29 – June 4 1999, also in J. McKay, *Proceedings on moonshine and related topics*, American Mathematical Society, Providence, RI, USA (2001), [math/9910136](#) [SPIRES].
- [15] G. Mason and M.P. Tuite, *The Genus Two Partition Function for Free Bosonic and Lattice Vertex Operator Algebras*, [arXiv:0712.0628](#) [SPIRES].
- [16] A.A. Belavin and V.G. Knizhnik, *Algebraic Geometry and the Geometry of Quantum Strings*, *Phys. Lett. B* **168** (1986) 201 [SPIRES].
- [17] L. Álvarez-Gaumé, G.W. Moore and C. Vafa, *Theta functions, modular invariance and strings*, *Commun. Math. Phys.* **106** (1986) 1 [SPIRES].
- [18] E.P. Verlinde and H.L. Verlinde, *Chiral bosonization, determinants and the string partition function*, *Nucl. Phys. B* **288** (1987) 357 [SPIRES].
- [19] E. D’Hoker and D.H. Phong, *The Geometry of String Perturbation Theory*, *Rev. Mod. Phys.* **60** (1988) 917 [SPIRES].
- [20] E. D’Hoker and D.H. Phong, *Lectures on Two-Loop Superstrings*, [hep-th/0211111](#) [SPIRES].
- [21] S. Albeverio, J. Jost, S. Paycha and S. Scarlatti, *A mathematical introduction to string theory*, volume **225** of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, U.K. (1997), pg. 1–135 [SPIRES].
- [22] S.L. Cacciatori, F. Dalla Piazza and B. van Geemen, *Modular Forms and Three Loop Superstring Amplitudes*, *Nucl. Phys. B* **800** (2008) 565 [[arXiv:0801.2543](#)] [SPIRES].
- [23] S. Grushevsky, *Superstring scattering amplitudes in higher genus*, *Commun. Math. Phys.* **287** (2009) 749 [[arXiv:0803.3469](#)] [SPIRES].
- [24] S. Grushevsky and R. Salvati Manni, *The vanishing of two-point functions for three-loop superstring scattering amplitudes*, [arXiv:0806.0354](#) [SPIRES].
- [25] M. Matone and R. Volpato, *Superstring measure and non-renormalization of the three-point amplitude*, *Nucl. Phys. B* **806** (2009) 735 [[arXiv:0806.4370](#)] [SPIRES].
- [26] P. Goddard, *Meromorphic conformal field theory*, in *Infinite dimensional Lie algebras and Lie groups: Proceedings of the CIRM Luminy Conference, 1988*, World Scientific, Singapore (1989) pg. 556 [SPIRES].
- [27] J.-I. Igusa, *On Siegel modular forms of genus two*, *Amer. J. Math.* **84** (1962) 175.

- [28] S. Tsuyumine, *On Siegel modular forms of degree three*, *Amer. J. Math.* **108** (1986) 755.
- [29] J.-I. Igusa, *Schottky's invariant and quadratic forms*, in *E.B. Christoffel. The influence of his work on mathematics and the physical sciences, International Christoffel Symposium*, Aachen and Monschau, Germany, November 8–11 1979, Birkhäuser Verlag, Basel (1981) pg. 352–362.
- [30] S. Böcherer, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen*, *Math. Z.* **183** (1983) 21.
- [31] C. Poor and D. Yuen, *Dimensions of spaces of Siegel modular forms of low weight in degree four*, *Bull. Austral. Math. Soc.* **54** (1996) 309.
- [32] A.W. Knap, *Lie groups beyond an introduction*, Progress in Mathematics **140**, Birkhäuser Boston Inc., Boston, MA, USA (2002).
- [33] S. Okubo, *Casimir Invariants and Vector Operators in Simple Lie Algebra*, *J. Math. Phys.* **18** (1977) 2382 [SPIRES].
- [34] J.D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics **352**, Springer-Verlag, Berlin, Germany (1973).